ABOUT THE AUTHOR

Brian is a young Mining Engineer who has an unwavering passion for education. His core specialty being Maths and Science. He has been tutoring for over 5 years and is the co-founder of a tutoring and publishing company Inspire Academy which sees an influx of students annually .He has tutored in Witwatersrand University, at the SIPP(Secondary School Improvement Program) and private clients ranging from Hilton College, St Teresa, Roedean Senior, and Mbilwi Secondary just to name a few.

Brian's core belief is that every student has the inert potential to achieve excellence in Maths and any other subject; they just need the guidance and support to reveal that potential. Through this passion Brian realized while tutoring that students struggled with the core basics and critical areas in Maths, hence, has written **Maths Simplified**.

Brian's aim with **Maths Simplified** is to deconstruct and simplify Maths to its core basics to allow learners to reach their fullest potential. It is easy to use and provides learners with a guide that when studied diligently and intentionally will fully prepare them for their exams.

ACKNOWLEDGMENTS

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This book is dedicated to all the courageous students I have taught, yet to teach and those who I will connect and touch with through this book. I wish you all the best in your academics and life, and remember, be kind, live to inspire and lift as you rise. I honor and salute you!

HOW TO STUDY WITH THIS BOOK

KEY TO COLOUR CODING

IMPORTANT	Key concepts to know and practice	Tip!	Useful advice
REMEMBER	Information to recall		Definitions and explanations
NOTE	Information to highlight		Key information and steps
→	The flow of calculations		Heading

1. Be aware of the different colours used and their meanings.

2. Find a comfortable place to study with good lighting and few distractions. Avoid studying in bed, as it will be too tempting to take a nap.

3. The best way to study is to work on each topic individually to understand the basics. Start by reading the chapter notes, followed by doing the worked examples before attempting the exam exercises at the end of each chapter.

4. Focus for between 40-90 minutes, then take a 10-minute break where you do exercise. Regular exercise helps with concentration.

5. The key to mastering mathematics is repetition! It is true that practice makes perfect. Practice by doing a sum or activity and note your problem areas with a pencil then retry the sum or activity. Master the basics!

6. Make sure to study the most difficult material when you most alert and do the easier topics when you are not quite as focused.

7. Work through the whole book without skipping any chapters, exam calculations and exam exercises.
   Meticulous care and diligence were taken to select the best exercises to prepare you best for the exams.

8. Understand and memorize the derivation of proofs and mathematical symbols.

9. Focus on the exam. Knowing the material is only part of it. Do as many mock exams and past papers as possible to practice time management and be comfortable with exam standards.

10. Lastly, have fun!

TABLE OF CONTENTS

ALGEBRA

1. TRINOMIALS

To solve trinomials, use the Quadratic equation:

$ax^2 + bx + c \quad [a, b, c \text{ are constants}]$ $\qquad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The ***plus***(+) and ***minus***(−) indicates that there are **two**(**2**)**solutions**.

Important Steps to solving equations:

1. **Transposing:** Every variable that moves across the equal sign changes its sign.

e.g. $x^2 = 4x + 8$

$x^2 = 4x + 8 \quad \rightarrow \quad x^2 - 4x - 8 = 0$

2. **Multiplying out brackets:** A variable outside a bracket multiplies all the variables in the bracket.

e.g. $x(x + 1)$

$= x(x + 1)$

$= x^2 + 1x = x^2 + x$

e.g. $(x + 1)(x - 1)$

$= (x + 1)(x - 1) \rightarrow = x^2 - x + x - 1 = x^2 - 1$

3. **Simplifying square roots:** Always isolate variables with the square root and simplify by squaring both sides of the equation.

e.g. $\sqrt{x + 5} = x + 3$

$\left(\sqrt{x + 5}\right)^2 = (x + 3)^2$

$x + 5 = (x + 3)(x + 3)$

$x + 5 = x^2 + 3x + 3x + 9$

$x + 5 = x^2 + 6x + 9$

$x + 5 = x^2 + 6x + 9 \rightarrow 0 = x^2 + 6x - x + 9 - 5$

$0 = x^2 + 5x + 4$

$x^2 + 5x + 4 = 0 \rightarrow (x + 4)(x + 1) = 0$

$x + 4 = 0 \ \ or \ \ x + 1 = 0$

$\therefore x = -4 \ or \ x = -1$

REMEMBER

A **square root** cancels a **square**, and vice verse.

NOTE

Always **validate** your answers! When validating we substitute our factors into the expression, if a factor is substituted into the expression and the **LHS** = **RHS** of the expression then that factor is the solution. $\therefore \sqrt{-4 + 5} = -4 + 3 \rightarrow \sqrt{1} = -1 \rightarrow 1 \neq -1 \therefore x = -4$ is not a solution

$\sqrt{-1 + 5} = -1 + 3 \rightarrow \sqrt{4} = 2 \rightarrow 2 = 2 \quad \therefore \boldsymbol{x = -1}$ **is a solution**.

Steps to solving quadratic equations

Step:1. Simplify equation : e.g. $(x-a)(x+b)=d$.

Step:2. Write equation in standard form: $ax^2+bx+c=0$.

Step:3. Use the standard form : $ax^2+bx+c=0$ to find the factors:$(x-a)(x+b)=0$ **OR** quadratic eq.

Step:4. Write out the solutions.

Example:1. Solve for x: $x^2-3x=4$.

Step:1. $x^2-3x=4 \rightarrow x^2-3x=4 \rightarrow x^2-3x-4=0$ |simplified|

Step:2. $x^2-3x-4=0$ |standard form|

Step:3. $x^2-3x-4=0 \rightarrow (x-4)(x+1)=0$ |factors|

Step:4. $(x-4)(x+1)=0 \rightarrow (x-4)=0 \ or \ (x+1)=0 \rightarrow \therefore x=4 \ or \ x=-1$ |answers|

Example:2. Solve for x: $x^2+4x+4=0$.

We are given the equation in standard form, ∴ we move on to **Step:3.**

Step:3. $x^2+4x+4=0 \rightarrow (x+2)(x+2)=0$ |factors|

Step:4. $(x+2)(x+2)=0 \rightarrow (x+2)=0 \ or \ (x+2)=0 \rightarrow \therefore x=-2 \ or \ x=-2$ |answers|

Example:3. Solve for x: $(x+2)(x-1)=0$.

We are given the factors,∴ we move on to **Step:4.**

Step:4. $(x+2)(x-1)=0 \rightarrow (x+2)=0 \ or \ (x-1)=0 \rightarrow \therefore x=-2 \ or \ x=1$ |answers|

Example:4. Solve for x: $x(4-x)=0$.

We are given the factors,∴ we move on to **Step:4.**

Step:4. $x(4-x)=0 \rightarrow x=0 \ or \ 4-x=0$

$x=0 \ or \ 4-x=0 \rightarrow x=0 \ or \ x=4$ |answers|

You can use a calculator to move from **Step:2** to **Step:4.**

IMPORTANT

Do not simplify the equation: $x(4-x)=0$

$x(4-x)=0 \rightarrow 4x-x^2=0$ this will move the equation to Step: 2.

Steps on using the quadratic equation on your calculator

1.Press: − **2.** Enter: − **3.**Enter: (**4.**Enter:value of b **5.**Enter:) **6.**Enter: $-$ *or* $+$ **7.** Press: $\sqrt{\ }$ **8.**Enter: (**9.**Enter: value of b **10.**Enter:) **11.** Press: x **12.**Enter: 2 then scroll to the right: press the → button **13.**Enter: − **14.**Enter: 4 **15.**Enter: value of a in brackets (a) **16.**Enter: × **17.**Enter: value of c in brackets (c) **18.**Scroll down:Press the ↓ button **19.**Enter: 2 **20.**Enter: value of c in brackets (c) **21.**Enter: = . Repeat the process but change the sign at Step:6 to find the solution of the other solution.

$$ax^2+bx+c=0 \qquad x_1=\frac{-b+\sqrt{b^2-4ac}}{2a} \qquad or \qquad x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

2. INEQUALITIES

Inequality signs

$<$: less than $>$: greater than $\leq$: less than or equal to $\geq$: greater than or equal to

The inequality sign changes on two conditions:

1.When multiplying or dividing throughtout by a negative sign.

Example: $-x \leq -4$

$\frac{-x}{-1} \leq \frac{-4}{-1} \quad \therefore x \geq 4$

2.When a number being transposed changes a sign from positive to negative.

Example: $x^2 - 25 \geq 0$

$(x-5)(x+5) \geq 0 \rightarrow x - 5 \geq 0 \ \ or \ \ x + 5 \geq 0$

$x \geq 5 \ \ or \ \ x \leq -5$

REMEMBER

The **inequality sign** changes direction when we **multiply** or **divide** by a **negative**$(-)$**sign**.

Steps to solving inequalities

Step:1. Simplify equation.

Step:2. Write equation in standard form.

Step:3. Use the standard form to find the factors **OR** use the quadratic equation.

Step:4. Write out the critical values in correct notation.

Example:1. Solve for x: $x^2 - 2x \leq 15$.

Step:1. $x^2 - 2x \leq 15 \ \rightarrow \ x^2 - 2x \leq 15 \rightarrow x^2 - 2x - 15 \leq 0$ |simplified|

Step:2. $x^2 - 2x - 15 \leq 0$ |standard form|

Step:3. $x^2 - 2x - 15 \leq 0 \rightarrow \ (x-5)(x+3) \leq 0$ |factors|

Step:4. $(x-5)(x+3) \leq 0 \rightarrow x - 5 \leq 0 \ or \ x + 3 \leq 0 \ \rightarrow \ x \leq 5 \ or \ x \geq -3$ |critical values|

Example:2. Solve for x: $16 - 4x \geq 0$.

Step:1. $16 - 4x \geq 0 \rightarrow 16 - 4x \geq 0 \ \rightarrow \ -4x \geq -16$

The expression is not quadratic, we therefore, move to Step:4.

Step:4. $-4x \geq -16 \ \rightarrow \ \frac{-4x}{-4} \geq -\frac{16}{-4} \quad \therefore x \leq 4$ |critical value|

3. EXPONENTIAL EQUATIONS

There are **THREE(3)** methods of solving exponential equations. Master all three!

1. **Changing the bases of exponents to be the same.**

$Solve\ for\ x$: $3^x.3^{x+1} = 27^x$

$3^x.3^{x+1} = 27^x \rightarrow 3^x.3^{x+1} = (3^3)^x \rightarrow 3^{x+x+1} = 3^{3x}$

$3^{2x+1} = 3^{3x} \rightarrow 2x + 1 = 3x \rightarrow 2x - 3x = -1 \rightarrow -x = -1 \ \therefore \boldsymbol{x = 1}$

Change 27 to have a base of 3 — Apply Exponential Law 7

NOTE

LAWS OF EXPONENTS

1. $a^n.a^m = a^{n+m}$
2. $\frac{a^n}{a^m} = a^{n-m}$
3. $(a^n.a^m)^c = a^{n\times c}.a^{m\times c}$
4. $\left(\frac{a^n}{b^m}\right)^c = \frac{a^{n\times c}}{b^{m\times c}}$
5. $a^{-n} = \frac{1}{a^n}$
6. $a^0 = 1$
7. $a^m = a^n \ \therefore m = n$

2. **Removing a common factor.**

$Solve\ for\ x$: $2^{x+2} + 2^x = 20$

$2^{x+2} + 2^x = 20 \rightarrow 2^x.2^2 + 2^x = 20 \rightarrow 2^x.2^2 + 2^x = 20$

$2^x(2^2 + 1) = 20 \rightarrow 2^x(4 + 1) = 20 \rightarrow 2^x(5) = 20 \rightarrow 2^x = \frac{20}{5}$

$2^x = 4 \rightarrow 2^x = 2^2 \ \therefore \boldsymbol{x = 2}$

Apply Exponential Law 1

2^x is a common factor

3. **Using the K-method of substitution.**

$Solve\ for\ x$: $2^{2x} + 6.2^x = 16$ Convert 2^{2x} to $(2^x)^2$

$2^{2x} + 6.2^x = 16 \rightarrow 2^{2x} + 6.2^x - 16 = 0 \rightarrow (2^x)^2 + 6.2^x - 16 = 0$ To simplify the equation

Let $2^x = k$, where there is 2^x we substitute k : $(k)^2 + 6(k) - 16 = 0 \rightarrow k^2 + 6.k - 16 = 0$

Option: 1

$k^2 + 6k - 16 = 0$

$(k + 8)(k - 2) = 0$

$k + 8 = 0 \quad or \quad k - 2 = 0$

$k = \boldsymbol{-8} \quad or \quad k = \boldsymbol{2}$

Substitute back $\boldsymbol{2^x}$ where there is $\boldsymbol{k}$

$\boldsymbol{2^x = -8} \quad or \quad \boldsymbol{2^x = 2^1}$

No Solution ***or*** $\boldsymbol{x = 1}$

OR

Option: 2

$k^2 + 6k - 16 = 0$

$a = 1, b = 6, c = -16$

$k = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$k = \frac{-(6) \pm \sqrt{(6)^2 - 4(1)(-16)}}{2(1)}$

$k = \boldsymbol{-8} \quad or \quad k = \boldsymbol{2}$

$\boldsymbol{2^x = -8} \quad or \quad \boldsymbol{2^x = 2^1}$

No Solution ***or*** $\boldsymbol{x = 1}$

EXAM CALCULATIONS ON FACTORIZATION,EXPONENTIAL EQUATIONS AND INEQUALITITES

1.

Solve for x

$x + \sqrt{x-2} = 4$ (5)

Interpretation:

$x + \sqrt{x-2} = 4$ → We need to ***eliminate*** the ***sqaure root by squaring both sides of the equation***. **NOTE: Before we square both sides of the equation we need to isolate the square root.**

$x + \sqrt{x-2} = 4 \rightarrow x + \sqrt{x-2} = 4 \rightarrow \sqrt{x-2} = 4 - x \rightarrow \left(\sqrt{x-2}\right)^2 = (4-x)^2$

$x - 2 = (4-x)(4-x) \rightarrow x - 2 = (4-x)(4-x) \rightarrow x - 2 = 16 + 4(-x) - x(4) - x \times (-x)$

$x - 2 = 16 - 4x - 4x + x^2 \rightarrow x - 2 = 16 - 8x + x^2 \rightarrow 0 = 18 - 9x + x^2 \rightarrow 0 = 18 - 9x + x^2$

$\boldsymbol{x^2 - 9x + 18 = 0} \rightarrow (x-3)(x-6) = 0 \rightarrow x - 3 = 0 \quad or \quad x - 6 = 0\,,\ x = 3 \quad or \quad x = 6$

we need to validate our values.

Tip!

Find the **factors** of **18** that which we **add** we get $-$ **9** and that which we **multiply** we $+$ **18**.

$x + \sqrt{x-2} = 4 \quad substitute\ 3$	$x + \sqrt{x-2} = 4 \quad substitute\ 6$
$3 + \sqrt{(3)-2} = 4$	$6 + \sqrt{(6)-2} = 4$
$3 + \sqrt{1} = 4$	$6 + \sqrt{4} = 4$
$3 + 1 = 4$	$6 + 2 = 4$
$4 = 4$	$8 \neq 4$

$\therefore$ $\boldsymbol{x = 3}$ **is the only solution**.

2.

Solve for x

1.1.4. $\left(\sqrt{\sqrt{32} + x}\right)\left(\sqrt{\sqrt{32} - x}\right) = x$ (4)

Interpretation:

$\left(\sqrt{\sqrt{32}+x}\right)\left(\sqrt{\sqrt{32}-x}\right)=x \qquad \sqrt{a}\times\sqrt{a}=a$

REMEMBER

$\sqrt{a}\times\sqrt{a}=a$

OR

$\sqrt{a}\times\sqrt{a}=a^{\frac{1}{2}}\times a^{\frac{1}{2}}=a^{\frac{1}{2}+\frac{1}{2}}$

$a^1=a$ and

$\sqrt{a}\times\sqrt{b}=\sqrt{a\times b}$

$\sqrt{\sqrt{32}+x}\times\sqrt{\sqrt{32}-x}=x \rightarrow \left(\sqrt{(\sqrt{32}+x)(\sqrt{32}-x)}\right)^2=(x)^2$

$(\sqrt{32}+x)(\sqrt{32}-x)=x^2 \rightarrow (\sqrt{32}+x)(\sqrt{32}-x)=x^2$

$\sqrt{32}\times\sqrt{32}+\sqrt{32}\times(-x)+x\times\sqrt{32}+x\times(-x)=x^2 \rightarrow 32-\sqrt{32}x+\sqrt{32}x-x^2=x^2$

$32-\sqrt{32}x+\sqrt{32}x-x^2-x^2=0 \rightarrow 32-2x^2=0 \rightarrow -2x^2=-32 \rightarrow \frac{-2x^2}{-2}=\frac{-32}{-2} \rightarrow x^2=16$

$x=\sqrt{16} \quad \therefore\ x=4 \quad or\ \ x=-4$

Validation

$\left(\sqrt{\sqrt{32}+4}\right)\left(\sqrt{\sqrt{32}-4}\right)=4 \rightarrow 4=4 \rightarrow \therefore\ \boldsymbol{x=4}$ **is a solution.**

$\left(\sqrt{\sqrt{32}-4}\right)\left(\sqrt{\sqrt{32}-(-4)}\right)=-4 \rightarrow 4=-4 \rightarrow \therefore\ x=-4$ is not asolution.

3.

Evaluate:

$\frac{3^{2018}+3^{2016}}{3^{2017}}$ (2)

Interpretation:

$\frac{3^{2018}+3^{2016}}{3^{2017}} \rightarrow 3^{2016}$ is the lowest exponent, we need to change all the exponents to the base of 3^{2016}.

$\frac{3^{2016}.3^2+3^{2016}}{3^{2016}.3^1}$ Common factor $\rightarrow \frac{3^{2016}.3^2+3^{2016}}{3^{2016}.3^1}$

$\frac{3^{2016}(3^2+1)}{3^{2016}.3^1} \rightarrow \frac{3^{2016}(3^2+1)}{3^{2016}.3^1} \rightarrow \frac{(9+1)}{3}=\boldsymbol{\frac{10}{3}}$

4.

Solve for x

1.1.5. $2^{x+2} + 7.\sqrt{2^x} = 2$ (5)

Interpretation:

$2^{x+2} + 7.\sqrt{2^x} = 2$ ⟶ Separate $2^{x+2} \rightarrow 2^x.2^2$

$2^{x+2} + 7.\sqrt{2^x} = 2 \rightarrow 2^x.2^2 + 7.\sqrt{2^x} = 2$ Let$2^x = k$ $k.2^2 + 7.\sqrt{k} = 2 \rightarrow 4k + 7.\sqrt{k} = 2 \rightarrow 7\sqrt{k} = 2 - 4k$

$\left(7\sqrt{k}\right)^2 = (2-4k)^2 \rightarrow 7^2.\left(\sqrt{k}\right)^2 = (2-4k)(2-4k) \rightarrow 49.k = (2-4k)(2-4k)$

$49k = 2 \times 2 + 2 \times (-4k) - 4k \times (2) - 4k \times (-4k) \rightarrow 49k = 4 - 8k - 8k + 16k^2$

$16k^2 - 65k + 4 = 0 \rightarrow (k-4)(16k-1) = 0 \rightarrow \quad k - 4 = 0 \quad or \quad 16k - 1 = 0 \quad \rightarrow k = 4 \quad or \quad k = \frac{1}{16}$

Replace k by 2^x $2^x = 4 \quad or \quad 2^x = \frac{1}{16} \rightarrow 2^x = 2^2 \quad or \quad 2^x = 2^{-4} \rightarrow x = 2 \quad or \quad x = -4$

Validation
$2^{2+2} + 7.\sqrt{2^2} = 2 \rightarrow 30 = 2 \rightarrow \therefore x = 2$ is not a solution.
$2^{-4+2} + 7.\sqrt{2^{-4}} = 2 \rightarrow 2 = 2 \rightarrow \therefore \boldsymbol{x = -4}$ **is a solution**.

5.

Solve for x

1.1.4. $x = \frac{a^2 + a - 2}{a - 1}$ if $a = 888\,888\,888\,888$ (2)

Interpretation:

$x = \frac{a^2 + a - 2}{a - 1}$

$x = \frac{a^2 + a - 2}{a - 1} \quad a = 888\,888\,888\,888 \quad x = \frac{(a+2)(a-1)}{a-1} \rightarrow x = a + 2$

$x = 888\,888\,888\,888 + 2 \quad \therefore \boldsymbol{x = 888\,888\,888\,890}$.

SIMULTANEOUS EQUATIONS

Simultaneous equations consist of two(2) unknown variables from two(2) equations.

Steps to solving simultaneous equations

Step:1. From the two(2) equations, label one equation, equation 1, and the other equation, equation 2.

Step:2. Construct(make) equation 3 from either equation 1 or equation 2.

Note: Always choose the less complex (or simple) equation to construct(make)equation 3.

TIP! One of the equations will have a variable of one(1), make that variable the subject of the equation, the coefficient of that variable should always be (+1).

Step:3. Substitute equation 3 into the used equation. Always put equation 3 in brackets when substituting it into the unused equation.

Step:4. Substitute the answer of the variable into equation 3 to find the other variable.

Step:5. Display the factors.

EXAM CALCULATIONS ON SIMULTANEOUS EQUATIONS

1.

Solve for x and y

$-2y + x = -1$ and $x^2 - 7 - y^2 = -y$ (6)

Interpretation:

$-2y + x = -1$ $x^2 - 7 - y^2 = -y$

Two equations and two unkowns, x and y.

NOTE

Variable: a symbol (usually a letter) representing an unknown numerical value in an equation. Commonly used variables are x and y.

Coefficient: a coefficint is a number multiplied by a variable.

$e.g. 32x$

32 is the coefficient and x is the variable.

$-2y + x = -1$ $\quad x^2 - 7 - y^2 = -y$

Step:1. Label equation 1 and 2.

$-2y + x = -1$ ——— (1) $\quad x^2 - 7 - y^2 = -y$ ——— (2)

Step:2. Construct equation 2 from either equation 1 or 2.

Note: Equation 1 is less complex and has no square variables and we have $+ x$.

notice, x and y cannot be made subject of the equation from equation 2 because of the squares.

$-2y + x = -1 \quad x = 2y - 1$ ——— (3)

Step:3. Substitute equation 3 into the used equation. Remember to put equation 3 into brackets.

Equation 2 was not used to construct equation 3, therefore, we substitute equation 3 into 2.

$x = 2y - 1$ ——— (3) $\quad x^2 - 7 - y^2 = -y$ ——— (2)

Where there is x in the equation 3 we substitute $2y - 1$.

$(2y - 1)^2 - 7 - y^2 = -y$

$(2y - 1)(2y - 1) - 7 - y^2 = -y$ ⟶ Note that there is only one(1) unknown varaible, y.

$(2y - 1)(2y - 1) - 7 - y^2 = -y$ Therefore, we solve for y.

$2y \times 2y + 2y \times (-1) - 1 \times (2y) - 1 \times (-1) - 7 - y^2 = -y$

$4y^2 - 2y - 2y + 1 - 7 - y^2 = -y$

$4y^2 - 4y - 6 - y^2 = -y$

$4y^2 - y^2 - 4y + y - 6 = 0$

$3y^2 - 3y - 6 = 0 \rightarrow \frac{3y^2}{3} - \frac{3y}{3} - \frac{6}{3} = 0 \rightarrow y^2 - y - 2 = 0 \rightarrow (y - 2)(y + 1) = 0$

$y - 2 = 0 \ \ or \ \ y + 1 = 0 \rightarrow \ \ y = 2 \quad or \quad y = -1$

Step:4. Substitute the answer of the variable into equation 3.

$y = 2$ $\qquad$ $y = -1$

$x = 2y - 1$ $\qquad$ $x = 2y - 1$

$x = 2(2) - 1$ $\qquad$ $x = 2(-1) - 1$

$x = 4 - 1$ $\qquad$ $x = -2 - 1$

$x = 3$ $\qquad$ $x = -3$

Step:5. Display the factors: $\quad \therefore \ \ \boldsymbol{x = 3, y = 2}$, and $\boldsymbol{x = -3, y = -1}$.

2.

Solve for x and y simultaneously in the following equations:

$x - 2y = 3$ and $4x^2 - 3 = -6y + 5xy$ (6)

Interpretation:

$x - 2y = 3$ $4x^2 - 3 = -6y + 5xy$ Two equations and two unknowns, x *and* y

---

Step:1. Label equation 1 and equation 2.

$x - 2y = 3$ ——— (1) $4x^2 - 3 = -6 + 5xy$ ——— (2)

Step:2. Construct equation 3 from either equation 1 or 2.

Note that equation 1 is the less complex equation, therefore, we choose equation 1 to contruct equation 3.

$x - 2y = 3$ ——— (1) $4x^2 - 3 = -6 + 5xy$ ——— (2)

$x = 3 + 2y$ ——— (3)

Step:3. Substitute equation3 into the unused equation

The unused equation is equation 2 therefore, substitute equation 3 into equation 2

$4(3 + 2y)^2 - 3 = -6 + 5(3 + 2y)y$

$4(3 + 2y)(3 + 2y) - 3 = -6 + 5(3 + 2y)y$

$4(3 + 2y)(3 + 2y) - 3 = -6 + 5y(3 + 2y)$

$4[3 \times 3 + 3 \times 2y + 2y \times 3 + 2y \times 2y] - 3 = -6 + 5y \times 3 + 5y \times 2y$

$4[9 + 6y + 6y + 4y^2] - 3 = -6 + 15y + 10y^2$

$4[9 + 12y + 4y^2] - 3 = -6 + 15y + 10y^2 \rightarrow 4[9 + 12y + 4y^2] - 3 = -6 + 15y + 10y^2$

$4 \times 9 + 4 \times 12y + 4 \times 4y^2 - 3 = -6 + 15y + 10y^2 \rightarrow 36 + 48y + 16y^2 - 3 + 6 - 15y - 10y^2 = 0$

$6y^2 + 39y + 33 = 0 \rightarrow \frac{6y^2}{3} + \frac{39y}{3} + \frac{33}{3} = 0 \rightarrow 2y^2 + 13y + 11 = 0 \rightarrow (y + 1)(2y + 11) = 0$

$y + 1 = 0$ *or* $2y + 11 = 0 \rightarrow y = -1$ *or* $y = -\frac{11}{2}$

Step:4. Substitute into equation 3.

$y = -1$ into $x = 3 + 2y \rightarrow x = 3 + 2(-1) \rightarrow x = 3 - 2 \rightarrow x = 1$

$y = -\frac{11}{2}$ into $x = 3 + 2y \rightarrow x = 3 + 2\left(-\frac{11}{2}\right) \rightarrow x = -8$ **Step:5.** $\boldsymbol{y = -1, x = 1}$ $\boldsymbol{y = -\frac{11}{2}, x = -8}$.

NATURE OF ROOTS

Roots are the $\boldsymbol{x - values}$ that we find when we equate a quadratic expression to zero: $\boldsymbol{ax^2 + bx + c = 0}$, then solve for x. Either by factorization, completing the square, or using the quadratic formula.

$$\text{Quadratic Equation: } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, $b^2 - 4ac$ is called a **Discriminant**$\boldsymbol{(\Delta b^2 - 4ac)}$.

Types of roots:

1. Real
2. Equal
3. Real and Unequal
4. Rational
5. Irrational
6. Non − real or Imaginary

Natural Numbers $\mathbb{N}$

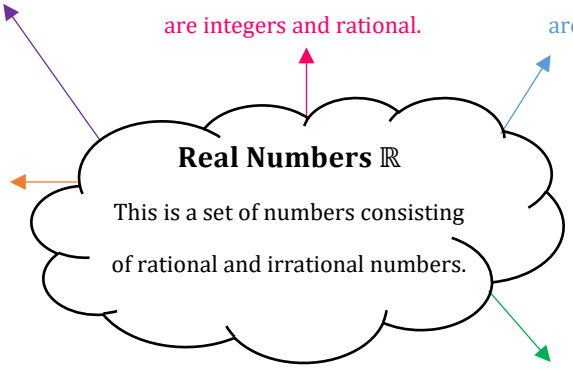
These are a set of counting of numbers: 1; 2; 3; 4 … starting from 1. These numbers are integers and rational.

Whole Numbers

These are a set of counting numbers: 0; 1; 2; 3; 4 … starting from 0. These numbers are integers and rational.

Intergers $\mathbb{Z}$

These are positive and negative counting numbers: $-2; -1; 0; 1; 2.$



Rational Numbers $\mathbb{Q}$

These are numbers writen in the form $\frac{a}{b}$, where a and b are intergers but $b \neq 0$.

e.g. $\frac{1}{2}, \frac{3}{4}$

Irrational Numbers

$\sqrt{\text{non} - \text{perfect squares}}$

$\sqrt{\text{non} - \text{perfect cubes}}$

e.g. $\sqrt{41}, \sqrt[3]{27}$

Non − real or Imaginary Numbers

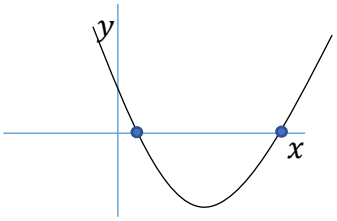
A square root of negative numbers

$\sqrt{\text{negative numbers}}$

e.g. $\sqrt{-4}$

Describing the nature of roots using the discriminant ($\Delta = b^2 - 4ac$) or graphically.

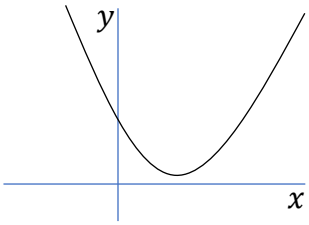
1. When the discriminant is greater than zero or positive($b^2 - 4ac > 0$) there will be **two real and unequal roots.**



Two real and unequal roots

$b^2 - 4ac > 0$ **two $x-$ intercepts**

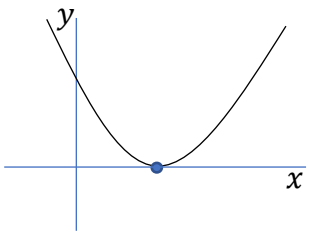
2. When the discriminant is less than zero or negative($b^2 - 4ac < 0$) there will be **no roots. The roots are non-real or imaginary.**



Roots are non $-$ real

$b^2 - 4ac < 0$ **no $x-$ intercepts**

3. When the discriminant is equal to zero($b^2 - 4ac = 0$) there will be **two real and equal roots.**



Two equal roots.

The turning point lies on the $x - axis$

$b^2 - 4ac = 0$ **$x-$ intercept = turning point**

4. When the discriminant is equal to a perfect square (a number whose square root is a whole number, e.g.$\sqrt{4} = 2, \sqrt{16} = 4$) there will be real and **rational roots.**

$$b^2 - 4ac = perfect\ square$$

5. When the discriminant is not equal to a non-perfect square (a number whose square root is not a whole number, e.g.$\sqrt{3}, \sqrt{23}$) there will be real and **irrational roots.**

$$b^2 - 4ac \neq perfect\ square$$

SUMMARY

Roots	Discriminant(Δ) : $b^2 - 4ac$
Real	$b^2 - 4ac \geq 0$
Equal	$b^2 - 4ac = 0$
Real and Unequal	$b^2 - 4ac > 0$
Rational	$b^2 - 4ac =$ perfect square
Irrational	$b^2 - 4ac \neq$ perfect square
Non-real/Imaginary	$b^2 - 4ac < 0$

Quadratic Equation: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ → This is the discriminant

Steps to approaching nature of roots.

Step:1. Write polynomial into a quadratic form: $ax^2 + bx + c$
Step:2. Choose the root(s) from the table (Nature of roots table as above) and the corresponding discriminant.
Step:3. Identify a, b, c from polynomial or quadratic equation.
Step:4. Substitute into the discriminant and solve for the unknown in question.

EXAM CALCULATIONS ON NATURE OF ROOTS

1.

For which value of k will the equation $x^2 + x = k$ have no real roots? (4)

Interpretation:

no real roots is the same as saying non − real roots, ∴ we use $b^2 - 4ac < 0$

Quadratic equation: $ax^2 + bx + c$

$x^2 + x = k \rightarrow x^2 + x - k = 0 \rightarrow a = 1, b = 1, c = -k \rightarrow$ no real roots, non − real : $b^2 - 4ac < 0$

$(1)^2 - 4(1)(-k) < 0 \rightarrow 1 - 4(-k) < 0 \rightarrow 1 + 4k < 0 \rightarrow 4k < -1 \rightarrow \dfrac{4k}{4} < -\dfrac{1}{4} \rightarrow \therefore \boldsymbol{k < -\dfrac{1}{4}}$.

2.

Determine the range of the function $y = x + \frac{1}{x}, x \neq$ and x is $real$. (6)

Interpretation:

Quadratic equation: $ax^2 + bx + c$

Re − arrange the equation: $y = x + \frac{1}{x} \rightarrow x.y = x.x + x.\frac{1}{x} \rightarrow xy = x^2 + 1 \rightarrow -x^2 + xy - 1 = 0$

$a = -1, b = xy, c = -1 \rightarrow$ Real: $b^2 - 4ac \geq 0 \rightarrow (y)^2 - 4(-1)(-1) \geq 0 \rightarrow y^2 - 4(1) \geq 0 \rightarrow y^2 - 4 \geq 0$

$y^2 - 2^2 \geq 0 \rightarrow (y-2)(y+2) \geq 0 \rightarrow y - 2 \geq 0 \quad or \quad y + 2 \geq 0 \rightarrow \mathbf{y \geq 2} \qquad \mathbf{or} \qquad \mathbf{y \leq -2}$

3.

Given: $f(x) = x^2 - 5x + 2$

For which values of c will $f(x) = c$ have no real roots? (4)

Interpretation:

No real roots is non − real roots: $b^2 - 4ac < 0$

NOTE

When we transpose a value over the inequality sign and it becomes a negative value, we invert the sign(the sign faces the opposite direction)

$f(x) = c \rightarrow x^2 - 5x + 2 = c \rightarrow x^2 - 5x + 2 - c = 0 \rightarrow a = 1, b = -5, c = 2 - c$

Non − real: $b^2 - 4ac < 0 \rightarrow (-5)^2 - 4(1)(2-c) < 0 \rightarrow 25 - 4(2-c) < 0$

$25 - 4 \times 2 - 4 \times (-c) < 0 \rightarrow 25 - 8 + 4c < 0 \rightarrow 17 + 4c < 0 \rightarrow 4c < -17 \rightarrow \frac{4c}{4} < -\frac{17}{4} \rightarrow \therefore \mathbf{c < -\frac{17}{4}}$

4.

Determine for which values of p will the equation $px^2 + (p-2)x + p = 0$ have equal roots. (5)

Interpretation:

$px^2 + (p-2)x + p = 0 \longrightarrow$ Quadratic equation: $ax^2 + bx + c$

$px^2 + (p-2)x + p = 0$

$a = p, b = (p-2), c = p$

$Equal: b^2 - 4ac = 0$

$(p-2)^2 - 4(p)(p) = 0$

$(p-2)(p-2) - 4p^2 = 0$

$(p-2)(p-2) - 4p^2 = 0$

$p \times p + p \times (-2) - 2 \times (p) - 2 \times (-2) - 4p^2 = 0$

$p^2 - 2p - 2p + 4 - 4p^2 = 0$

$p^2 - 4p^2 - 4p + 4 = 0$

$-3p^2 - 4p + 4 = 0$

$\frac{-3p^2}{-1} - \frac{4p}{-1} + \frac{4}{-1} = \frac{0}{-1}$

$3p^2 + 4p - 4 = 0$

$(3p-2)(p+2) = 0$

$3p - 2 = 0 \quad or \quad p + 2 = 0$

$3p = 2, \boldsymbol{p = \frac{2}{3}} \quad \boldsymbol{or} \quad \boldsymbol{p = -2}$

5.

Prove that the equation $6x^2 + 2px - 3x - p = 0$ has rational roots for all rational values of p. (4)

Interpretation:

$6x^2 + 2px - 3x - p = 0 \rightarrow 6x^2 + (2p-3)x - p = 0$ quadratic

---

$\Delta = b^2 - 4ac \qquad 6x^2 + (2p-3)x - p = 0 \qquad a = 6, b = (2p-3), c = -p$

$\Delta = (2p-3)^2 - 4(6)(-p) \rightarrow \Delta = (2p-3)(2p-3) + 24p \rightarrow \Delta = 4p^2 - 6p - 6p + 9 + 24p$

$\Delta = 4p^2 + 12p + 9 \rightarrow \Delta = (2p+3)(2p+3) \rightarrow \Delta = (2p+3)^2$

Rational roots: $b^2 - 4ac =$ perfect square, and $\Delta = (2p+3)^2$ is a perfect square. $\therefore$ **Rational roots**

6.

Prove that the roots of $a^2x^2 + abx + b^2 = 0$ are non $-$ real values of a and $b, a \neq 0$ and $b \neq 0$. (3)

Interpretation:

For the roots to be non $-$ real the dicriminant(Δ) needs to be < 0

$a^2x^2 + abx + b^2 = 0 \rightarrow \Delta = b^2 - 4ac \rightarrow a = a^2, b = ab, c = b^2 \rightarrow \Delta = (ab)^2 - 4(a^2)(b^2)$

$\Delta = a^2.b^2 - 4a^2, b^2 \rightarrow \Delta = -3a^2.b^2 \quad a \neq 0$ and $b \neq 0, \therefore a \times b > 0 \quad \boldsymbol{\Delta < 0, roots\ are\ non - real.}$

CHAPTER:1

EXAM EXERCISES ON ALGEBRA, SIMULTANEOUS EQUATIONS AND NATURE OF ROOTS

Exercise:1. **ALGEBRA**

Solve for x:

1.1. $x^2 - 9x + 20 = 0$ (3)

1.2. $3x^2 + 5x = 4$ correct to TWO decimal places (4)

1.3. $\sqrt{2-x} = x - 2$ (4)

1.4. $x^2 + 7x < 0$ (3)

1.5. $2x^{\frac{-5}{3}} = 64$ (4)

1.6. $(x-3)(x+1) = 5$ (3)

1.7. $9^{2x-1} = \frac{3^x}{3}$ (3)

1.8. $2\sqrt{2-7x} = \sqrt{-36x}$ (3)

1.9. $\frac{4x}{2} - \frac{2x+1}{3} = 5$ (2)

1.10. $(x-5)(x-6) \leq 56$ (5)

1.11. $5^{-x}.5^{x-2} = \frac{25^{2x}}{5}$ (4)

1.12. $(x-1)^2 = 2(1-x)$ (4)

1.13. $2^{2x} + 6.2^x = 16$ (4)

1.14. $\frac{3^{2015} + 3^{2013}}{9^{1006}}$ (3)

1.15. $2^{x+2} + 2^{x-2} + 2^x = 42$ (4)

1.16. $2x^2 - 2 \leq 3x$ (4)

1.17. $x(5x-1) = 2(7x+1)$ (4)

1.18. $3x + \frac{1}{x} = 4$ (4)

1.19. $4x + \frac{4}{x} + 11 = 0$ (4)

1.20. $3^{x^2-1} = \frac{27^{-x}}{3}$ (4)

Exercise:2. **SIMULTANEOUS EQUATIONS**

Solve for x and y in the following equations:

2.1. $x - 2y = 3$ and $4x^2 - 3 = -6y + 5xy$ (6)

2.2. $x + y - 2 = 0$ and $x(x - y + 1) = 0$ (6)

2.3. $x - y = 3$ and $x^2 + xy - 2y = 0$ (5)

2.4. $x + 3y = 2$ and $x^2 + 4xy - 5 = 0$ (5)

2.5. $3x - y = 4$ and $x^2 + 2xy - y^2 = -2$ (6)

2.6. $x = 2y + 2$ and $x^2 - 2xy + 3y^2 = 4$ (6)

2.7. $3^{x-10} = 3^{3x}$ and $y^2 + x = 20$ (5)

2.8. $3^{x+2} - 3^x = 8.9^y$ and $2x^2 = 2 - y^2$ (7)

2.9. $6r + 5rp - 5p = 8$ and $r + p = 2$ (7)

2.10. $2x + y = 3$ and $x^2 + y + x = y^2$ (8)

Exercise:3. **NATURE OF ROOTS**

3.1. $f(x) = \sqrt{(x-1)(x+2)}$

For which value(s) of x will f(x) be non $-$ real? (3)

3.2. Prove that the roots of

$2x^2 - (k-1)x + k - 3 = 0$ are real for all real values of k. (5)

3.3. $x = \frac{-2 \pm \sqrt{13 - 2k}}{3}$

Determine the largest value of k for which these $x-$ values will be real. (3)

3.4. Given: $f(x) = x^2 + 8x + 16$

For which values of p will $f(x) = p$ have TWO unequal negative roots? (4)

3.5. For which values of x will $\frac{\sqrt{3x-5}}{x-3}$ be real? (3)

3.6. $x = \frac{3 \pm \sqrt{-k-4}}{2}$ For which values of k are the roots real? (2)

3.7. For which value of k will the equation $x^2 + x = k$ have no real roots. (4)

3.8. Prove that the roots of the equation

$x^2 + (2-p)x - 3 - p = 0$ are real and unequal for all real values of p. (4)

3.9. Dermine the range of the function $y = x + \frac{1}{x}, x \neq 0$ and x is real. (6)

3.10. Given :

$f(x) = x^3 - 4x^2 - 2x + 20 = (x+2)(x^2 - 6x + 10)$.

Prove that $f(x)$ has only one real root. (3)

SEQUENCES

1. **ARITHMETIC SEQUENCE**
2. **QUADRATIC SEQUENCE**
3. **GEOMETRIC SEQUENCE**

1. ARITHMETIC SEQUENCE

An arithmetic pattern is a pattern that has a common difference between the terms. Consider the sequence: 2; 4; 6; 8; 10 , in this sequence to get the following term 2 is added to the previous term. This sequence is called an arithmetic sequence.

The formula representing the arithmetic sequence is called the **general term, n^{th} term or Tn.**

$General\ term$

$n^{th}\ term$

Tn

$$Tn = a + (n-1) \times d$$

$Tn = a + (n-1) \times d$

- Tn: the value of the n position. ($e.g.$ $the\ 5^{th} term\ is\ 40$)
- a: the first term, (T_1) of the sequence
- n: the position of the term ($e.g.\ 4^{th}\ term$)
- d: common difference

a: The first term of the sequence. (T_1)

d: The common difference, calculated by: $T_2 - T_1 = T_3 - T_2$

Example.

Consider the sequence: 2 ; 4 ; 6 ; 8.

$a = 2$

2 ; 4 ; 6 ; 8

2 2 2

$d = 2$

2. QUADRATIC SEQUENCE

A quadratic sequence is a sequence that has two differences, the first difference and a second difference. The second difference of a quadratic sequence is an arithmetic sequence.

The formula representing the quadratic sequence is called the **general term, n^{th} term or Tn**

General term

n^{th} *term*

Tn

$Tn = an^2 + bn + c$

Quadratic Sequence Example:

	5	7	11	17	25
1^{st} difference	2	4	6	8	
2^{nd} difference	2	2	2		

Notice that the second difference an arithmetic sequence.

	$n = 1$	$n = 2$	$n = 3$
1^{st} term	$a + b + c$	$4a + 2b + c$	$9a + 3b + c$
1^{st} difference	$3a + b$	$5a + b$	$7a + b$
2^{nd} difference		$2a$	$2a$

1^{st} term $= a + b + c$

1^{st} difference $= 3a + b$

2^{nd} difference $= 2a$

Steps to finding a, b, and c in $T_n = an^2 + bn + c$.

Step:1. Finding a.

$2a = 2^{nd}$ difference

Step:2. Finding b.

$3a + b = 1^{st}$ difference (use the a found in step 1)

Step:3. Finding c.

$a + b + c = 1^{st}$ term (use a and b from step 1 and 2)

REMEMBER

Always check the T_n, *General term or* n^{th} *term* by substituting any of the term numbers on the given sequence.

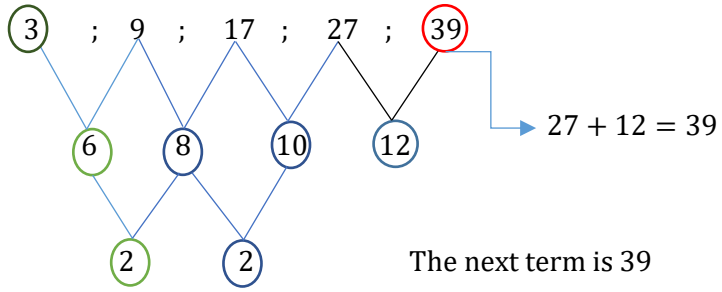
EXAM CALCULATIONS ON QUADRATIC SEQUENCE

1. The sequence 3; 9; 17; 27 …. is a quadratic sequence
1.1. Write down the next term
1.2. Determine the n^{th} term
1.3. Determine the term of the sequence greater than 269

Interpretation:

3; 9; 17; 27 ⟶ The quadratic sequence has two differences.

1.1.



The next term is 39

1.2.

The n^{th} term $T_n = an^2 + bn + c$

Step: 1

$2a = 2 \rightarrow a = \frac{2}{2} \rightarrow a = 1$

Step: 2

$3a + b = 6 \rightarrow 3(1) + b = 6$

$3 + b = 6 \rightarrow b = 6 - 3 \rightarrow b = 3$

Step: 3

$a + b + c = 3 \rightarrow 1 + 3 + c = 3$

$4 + c = 3 \rightarrow c = 3 - 4 \rightarrow c = -1$

$\therefore T_n = an^2 + bn + c$, $\boldsymbol{T_n = 1n^2 + 3n - 1}$.

1.3.

term $-$ n

greater than $> T_n$

$T_n = n^2 + 3n - 1$

$269 < n^2 + 3n - 1$ Solve for n

$n^2 + 3n - 1 - 269 > 0$

$n^2 + 3n - 270 > 0$

$(n + 18)(n - 15) > 0$

$n + 18 > 0$ *or* $n - 15 > 0$

$n < -18$ *or* $n > 15 \therefore \boldsymbol{n > 15}$ **because *n* cannot be negative**.

3. GEOMETRIC SEQUENCE

A geometric sequence is a sequence that has a common ratio. The common ratio is calculated by: $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

The formula representing the geometric sequence is called the **general term, n^{th} term or Tn**

$General\ term$

$n^{th}\ term$ $\quad Tn = a.(r)^{n-1}$

Tn

$$Tn = a.(r)^{n-1}$$

a is the first term

r is the common ratio

common ratio: $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

Using logarithms to find n.

$Tn = a.(r)^{n-1}$

$\frac{T_n}{a} = (r)^{n-1}$

$\log_r\left(\frac{T_n}{a}\right) = n - 1$

$n - 1 = \log_r\left(\frac{T_n}{a}\right)$

$\boldsymbol{n = \log_r\left(\frac{T_n}{a}\right) + 1}$

REMEMBER

$y = a^x$

$\log_a y = x$

e.g. $\quad 2 \ ; \ 4 \ ; \ 8 \ ; \ 16$

$\frac{4}{2}$ $\quad$ $\frac{8}{4}$ $\quad$ $\frac{16}{8}$

2 $\quad$ 2 $\quad$ 2

$a = 2$

common ratio $(r) = 2$

$\therefore\ T_n = a.(r)^{n-1}$

$T_n = 2.(2)^{n-1}$

REMEMBER

Always check the T_n, $General\ term\ or\ n^{th}\ term$ by substituting any of the term numbers on the given sequence.

EXAM CALCULATIONS ON GEOMETRIC SEQUENCE

1. Determine the general term of the geometric sequence: 2 ; 8 ; 32

$\boldsymbol{a = 2}$

$r = \frac{T_2}{T_1} = \frac{T_3}{T_2}$

$r = \frac{8}{2} = \frac{32}{8}$

$r = 4 = 4$

$\boldsymbol{r = 4} \quad \therefore \quad \boldsymbol{T_n = 2.(4)^{n-1}}$

$T_n = a.(r)^{n-1}$

CHECK.

$T_2 = 2.(4)^{2-1}$

$T_2 = 2.(4)^1$

$T_2 = 8$

✓ *CHECKED!*

2. Determine the 10^{th} term of the following sequence: 1 ; 5 ; 25.

$\boldsymbol{a = 1}$

$r = \frac{T_2}{T_1} = \frac{T_3}{T_2}$

$r = \frac{5}{1} = \frac{25}{5}$

$\boldsymbol{r = 5}$

$T_n = a.(r)^{n-1}$

$\therefore \boldsymbol{T_n = 1.(5)^{n-1}}$

$T_{10} = 1.(5)^{10-1}$ The 10^{th} term of the sequence is: $\boldsymbol{T_{10} = 1953125}$

3. Which term of the geometric sequence : 6 ; 3 ; $\frac{3}{2}$; will be equal to $\frac{3}{60}$.

Term is n

$\boldsymbol{a = 6}$

$\boldsymbol{T_n = \frac{3}{60}}$

$r = \frac{T_2}{T_1} = \frac{T_3}{T_2}$

$r = \frac{3}{6} = \frac{\frac{3}{2}}{3}$

$r = \frac{1}{2} = \frac{1}{2}$

$r = \frac{1}{2}$

$T_n = a.(r)^{n-1}$

$\therefore \boldsymbol{T_n = 6.\left(\frac{1}{2}\right)^{n-1}}$

$\frac{3}{60} = 6.\left(\frac{1}{2}\right)^{n-1} \rightarrow \frac{\frac{3}{60}}{6} = \left(\frac{1}{2}\right)^{n-1} \rightarrow \frac{1}{120} = \left(\frac{1}{2}\right)^{n-1} \rightarrow n - 1 = \log_{\frac{1}{2}} \frac{1}{120} \rightarrow n = \log_{\frac{1}{2}} \frac{1}{120} + 1$

$n = 7 + 1$

$\therefore \boldsymbol{n = 8}$.

4. THE COMBINATION OF THE ARITHMETIC AND GEOMETRIC SEQUENCE

EXAM CALCULATIONS ON THE COMBINATION OF THE ARITHMETIC AND GEOMETRIC SEQUENCE

1.

The following sequence has the property that the sequence of numerators is arithmetic and the sequence of denominators is geometric.

$$\frac{2}{1}\ ;\ -\frac{1}{5}\ ;\ -\frac{4}{25}\ldots$$

1. Write down the FOURTH term of the sequence. (1)

2. Determine a formula for the n^{th} term. (3)

3. Determine the 500^{th} term of the sequence. (2)

4. Which will be the first term of the sequence to have a NUMERATOR which is less than -59? (3)

Interpretation:

$Fraction = \frac{\text{Numerator}}{\text{Denominator}}$

1.

$2\ ;\ -1\ ;\ -4\ ;\ -7$ → Arithmetic sequence

$-3 \quad -3 \quad -3$

$a = 2 \quad d = -3$

$1\ ;\ 5\ ;\ 25$ → Geometric sequence

$5 \quad 5$

$a = 1 \quad r = 5$

$$\frac{2}{1}\ ;\ -\frac{1}{5}\ ;\ -\frac{4}{25}\ ;\ \boxed{-\frac{7}{125}}$$

2.

Arithmetic sequence:

$a = 2\,, d = -3$

$T_n = a + (n-1).d \rightarrow T_n = 2 + (n-1).(-3)$

$T_n = 2 + (-3n) + 3 \rightarrow T_n = -3n + 5$

$n^{th}\ term = \frac{Arithmetic\ T_n}{Geometric\ T_n} \rightarrow n^{th}\ term = \frac{-3n+5}{(5)^{n-1}}$

Geometric sequence:

$a = 1\,, r = 5$

$T_n = a.(r)^{n-1} \rightarrow T_n = 1.(5)^{n-1}$

$T_n = (5)^{n-1}$

$CHECK!\ T_2 = \frac{-3(2)+5}{(5)^{2-1}} = -\frac{1}{5}$ $CHECKED$

3.

$$500^{th}\ is\ T_{500} \rightarrow T_n = \frac{-3n+5}{(5)^{n-1}} \rightarrow T_{500} = \frac{-3(500)+5}{(5)^{500-1}} \rightarrow T_{500} = -\frac{1495}{5^{499}}$$

4.

NUMERATOR which is less than$(<) -59$

$$T_n = -3n+5 \rightarrow -3n+5 < -59 \rightarrow -3n < -59-5 \rightarrow -3n < -64 \rightarrow n < \frac{-64}{-3} \rightarrow n > \frac{64}{3}$$

2. Given the combined arithmetic and constant sequence:

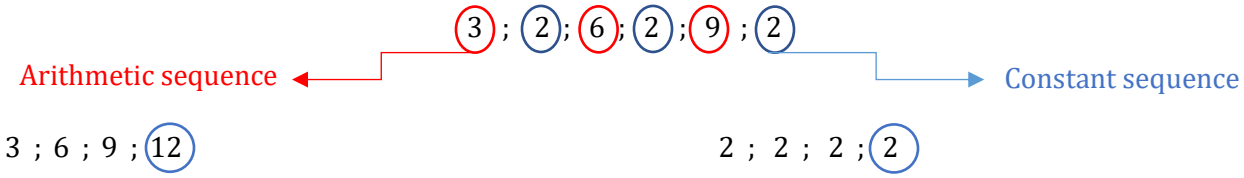
$$3\ ;\ 2\ ;\ 6\ ;\ 2\ ;\ 9\ ;\ 2\ ...$$

2.1. Write down the next two terms in the sequence.

2.2. Determine the 100^{th} term of the sequence.

Interpretation:

2.1.



The next two term of the sequence are : 12 ; 2

2.2.

The 100^{th} term is an even number. Which sequence is on the even position.

$1^{st}\ 2^{nd}\ 3^{rd}\ 4^{th}\ 5^{th}\ 6^{th}$

3 ; 2 ; 6 ; 2 ; 9 ; 2

therefore, the 100^{th} term is 2.

The constant(geometric) sequence is on the even number position therefore, 100 lies on the geometric sequence.

NOTE: Since there are 2 sequences the 100^{th} term will be the 50^{th} term of the geometric sequence.

$$a = 2\ , r = 1$$

$$therefore, T_n = a.(r)^{n-1}$$

$$T_n = 2.(1)^{n-1}$$

$$T_{50} = 2.(1)^{50-1}$$

$$T_{50} = 2$$

EXAM CALCULATIONS ON THE APPLICATION OF SEQUENCES

1.

Chris bought a Bonsai(miniature tree) at a nursery. When he bought the tree, it's height was 130mm.

Thereafter, the height of the tree increased, as shown below.

Increase in height of the tree per year		
During the first year	During the second year	During the third year
100mm	70mm	49mm

1. Chris noted that the sequence of height increases, namely: 100 ; 70 ; 49 ; … . was geometric.

During which year will the height of the tree increase by approximately11,76mm?

Interpretation:

Geometric sequence: 100 ; 70 ; 49 Therefore, $r = \frac{T_2}{T_1} = \frac{T_3}{T_2}$

---

100 ; 70 ; 49

$a = 100$

$r = \frac{T_2}{T_1} = \frac{T_3}{T_2}$

$r = \frac{70}{100} = \frac{49}{70}$

$r = \frac{7}{10} = \frac{7}{10}$

$T_n = a.(r)^{n-1}$

$T_n = 100.\left(\frac{7}{10}\right)^{n-1}$

The height of the tree increase

is 11, 76mm, therefore, $T_n = 11,76m$

$11,76 = 100.\left(\frac{7}{10}\right)^{n-1}$

$\frac{11,76}{100} = \left(\frac{7}{10}\right)^{n-1}$

$\log_{\left(\frac{7}{10}\right)}\left(\frac{11,76}{100}\right) = n - 1$

$n - 1 = \log_{\left(\frac{7}{10}\right)}\left(\frac{11,76}{100}\right)$

$n = \log_{\left(\frac{7}{10}\right)}\left(\frac{11,76}{100}\right) + 1$

$n = 6 + 1$

$n = 7$

It will take 7 year for the tree to reach

a the height of approximately 11, 76mm.

SERIES

The **addition** or **sum** of any sequence is called a series.

e. g. Sequence: 2 ; 4 ; 6 ; 8 Series: $2 + 4 + 6 + 8$

The sum of n terms is equal to the addition of n terms. The sum$(S_4) = 2 + 4 + 6 + 8 = 20$. To calculate larger term numbers we use formulae to simplify our work.

There are sums of arithmetic and geometric series.

1. THE SUM OF AN ARITHMETIC SERIES

a. $\boldsymbol{S_n = \frac{n}{2}[2a + (n-1) \times d]}$

$S_n - Sum\ of\ n\ terms$
$n - number\ of\ terms$
$a - first\ term$
$d - common\ difference$

b. $\boldsymbol{S_n = \frac{n}{2}(a + l)}$

$S_n - Sum\ of\ n\ terms$

$n - number\ of\ terms$

$l - last\ term$

$a - first\ term$

In general: $S_n = T_1 + T_2 + T_3 + T_4 + \ldots + T_n$
$\therefore S_1 = T_1 \ , \ S_2 = T_1 + T_2 \ , \quad S_3 = T_1 + T_2 + T_3$ *and* $S_4 = T_1 + T_2 + T_3 + T_4$

$S_4 = \boldsymbol{T_1 + T_2 + T_3} + T_4 \quad \rightarrow \quad S_4 = S_3 + T_4 \quad \therefore \quad T_4 = S_4 - S_3 \qquad \therefore \boldsymbol{T_n = S_n - S_{n-1}}$

$T_n = S_n - S_{n-1}$ is used to calculate terms when we given the sum formula. See exam question 4.

Proof of the sum formulas of the arithmetic series. *Examinable!*

$S_n = T_1 + T_2 + T_3 + T_4 \ldots\ldots + T_n \quad \rightarrow \quad \therefore S_n = a + (a + d) + (a + 2d) + \ldots\ldots + [a + (n-1)d]$

Let L(last term) $= T_n = [a + (n-1)d] \quad \therefore S_n = a + (a + d) + (a + 2d) + \ldots\ldots + L - d + L$ (1)

Re − write the equation into: $S_n = L + (L - d) + \ldots\ldots + (a + 2d) + (a + d) + a$ (2)

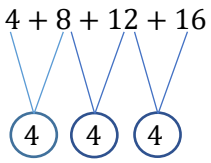
We add equation 1 and equation 2: $2S_n = (a + L) + (a + L) + \ldots\ldots + (a + L) + (a + L)$

$\therefore 2S_n = n(a + L) \quad \rightarrow \quad \therefore \quad S_n = \frac{n(a + L)}{2} \quad \rightarrow \quad S_n = \frac{n}{2}(a + L)$

$Substitute\ L\ into\ S_n = \frac{n(a + L)}{2} : \ S_n = \frac{n(a + a + (n-1)d)}{2} \rightarrow \therefore \ S_n = \frac{n(2a + (n-1)d)}{2}$

Example: **1**.

Calculate the sum of the arithmetic series : $4 + 8 + 12 + 16 + ..$ to 30 terms



$a = 4$

$d = 4$

$n = 30\ terms$

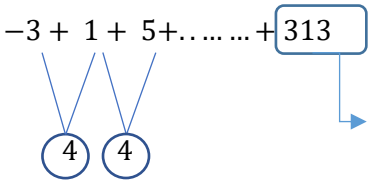
$$\boldsymbol{S_n = \frac{n}{2}[2a + (n-1) \times d]}$$

$$S_{30} = \frac{30}{2}[2 \times 4 + (30-1) \times 4]$$

$$S_{30} = 15[8 + 29 \times 4] \rightarrow \boldsymbol{S_{30} = 1860}$$

Example: **2**.

Calculate the sum of the arithmetic series : $-3 + 1 + 5 + ... + 313$



$a = -3 \rightarrow d = 4 \rightarrow l = 313 \rightarrow n = ?$

Tip!

We need to calculate n first before we calculate the sum of the sequence, n is the position of the last term 313. The position of the last term tells us how many terma are in a sequence

This is an arithmetic sequence, therefore, $T_n = a + (n-1).d$

$313 = -3 + (n-1) \times 4 \rightarrow 313 = -3 + 4n - 4 \rightarrow 313 + 3 + 4 = 4n \rightarrow 320 = 4n \rightarrow 4n = 320$

$n = \frac{320}{4} \rightarrow n = 80$ We can now calculate the sum of the sequence.

$a = -3$

$d = 4$

$l = 313$

$n = 80$

$$\boldsymbol{S_n = \frac{n}{2}(a + l)}$$

$$S_{80} = \frac{80}{2}(-3 + 313)$$

$$S_{80} = 40(310)$$

$$\boldsymbol{S_{80} = 12400}$$

2. THE SUM OF A GEOMETRIC SERIES AND SUM TO INFINITY

a. The Sum: $\boldsymbol{S_n = \frac{a(1-r^n)}{1-r}}$ **or** $\boldsymbol{S_n = \frac{a(r^n-1)}{r-1}}$

If r is less than one, $r < 1$.

Use the above equation

If r is greater than one, $r > 1$.

Use the above equation

Proof of the sum formulas of the geometric series. *Examinable*!

$S_n = T_1 + T_2 + \ldots\ldots + T_{n-1} + T_n \rightarrow \therefore S_n = a + ar + \ldots\ldots ar^{n-2} + ar^{n-1}$ (1)

To obtain equation 2 we multiply equation 1 by r.

$\therefore rS_n = ra + ar^2 + \ldots\ldots ar^{n-1} + ar^n$ (2)

Then equation 2 minus equation 1.

$rS_n - S_n = -a + ar^n$ simplify $S_n(r-1) = a(-1+r^n) \rightarrow S_n(r-1) = a(r^n-1) \therefore S_n = \frac{a(r^n-1)}{r-1}$

OR

equation 1 minus equation 2

$S_n - rS_n = a - ar^n$ simplify $S_n(1-r) = a(1-r^n) \rightarrow \therefore S_n = \frac{a(1-r)}{1-r}$

These formulae are used to calculate the sum total of terms or growth to a specific n.

b. Sum to infinity $\boldsymbol{S_\infty = \frac{a}{1-r}, r \neq 1}$ a = first term n – number of terms

An infinite series has an infinite number of terms. r = constant ratio S_n – Sum of n terms

This formula used to calculate the growth or maxima of a geometric series to an unknown n or number.

Convergent Series	**Divergent Series**
The series approaches a specific number.	The series approaches no specific number.

For a series to **converge**, the common ratio(r) needs to be greater than -1 and less than 1.

Represented as : $\boldsymbol{-1 < r < 1}$

If r does not fall between -1 and 1 then the series is **divergent**.

EXAM CALCULATIONS ON ARITHMETIC,GEOMETRIC AND SUM TO INFINITY

1. Given the geometric series : $\frac{24}{x} + 12 + 6x + 3x^2 + ..$

1.1. If $x = 4$, then determine the sum of 15 terms of the sequence.

1.2. Determine the values of x for which the original series converges.

Interpretation:

$\frac{24}{x} + 12 + 6x + 3x^2 + .$ $x = 4$, we subtitute x into the series $\frac{24}{4} + 12 + 6(4) + 3(4)^2 + ..$

$6 + 12 + 24 + 48 + ..$

1.1.

$6 + 12 + 24 + 48 + ..$

$a = 6$

$n = 15$

$r = \frac{T_2}{T_1} = \frac{T_3}{T_2}$, $r = \frac{12}{6} = \frac{24}{12}$, $r = 2$

NOTE: $r > 1$. Therefore, we use: $\boldsymbol{S_n = \frac{a(r^n - 1)}{r - 1}}$

$S_n = \frac{a(r^n - 1)}{r - 1} \rightarrow S_{15} = \frac{6(2^{15} - 1)}{2 - 1} \rightarrow S_{15} = \frac{6(32768 - 1)}{1}$

$\boldsymbol{S_{15} = 196602}$

1.2.

Converges: $\boldsymbol{-1 < r < 1}$

$\frac{24}{x} + 12 + 6x + 3x^2 + ..$

We need to calculate r.

$r = \frac{T_2}{T_1} = \frac{T_3}{T_2}$, $r = \frac{\frac{12}{24}}{\frac{x}{}} = \frac{6x}{12}$, $r = \frac{x}{2} = \frac{x}{2}$, $r = \frac{x}{2}$

$\frac{12}{\frac{24}{x}} = 12 \div \frac{24}{x} = 12 \times \frac{x}{24} = \frac{x}{2}$

To find x we substitute r and solve for x $\boldsymbol{-1 < r < 1}$

$-1 < \frac{x}{2} < 1$

$2. -1 < 2.\frac{x}{2} < 2.1$

$-2 < x < 2$, $x < 2$ *or* $x > -2$.

SIGMA NOTATION

$\sum$ – **this symbol indicates sigma**.

A sum of a series may be written using a summation symbol. The sigma notation indicates that you must find **the sum of the expression to the right of it**.

$$\sum_{n=start}^{end} T_n$$

Number of terms $= (end - start) + 1$

$n = (end - start) + 1$

Steps to evalutating sigma notation:

Step:1. Establish what series you are working with, Arithmeric or Geometric series.

Find the first three terms and find the difference or common ratio.

Note: You may be required find the sum of a quadratic sequence, and for this you need to manually add all the term values to find the sum.

Step:2. Find how many terms are in the series by:

$$\text{Number of terms}(n) = (End - Start) + 1$$

Step:3. Use the correct sum formula to find the sum

Step:4. Use $T_n = S_n - S_{n-1}$, to find the term value when you have the sum formula.

Steps to writing an expression in sigma notaion:

Step:1. Determine the general term of the series.

Step:2. Determine the number of terms in the series.

Step:3. Write the sigma notation of the expression in the form:

$$\sum_{n=start}^{end} T_n$$

EXAM CALCULATIONS ON SIGMA NOTATIONS

1. 2010 November DBE

Paper: 1 Q2

Evalute

$$\sum_{n=1}^{20} 3^{n-2} \qquad (4)$$

Interpretation:

We need to calculate the sum, by determining what series the above notation is arithmetic or geometric.

Step:1. We need to establish which series this is, Arithmetic or Geometric by finding the first three terms then the common difference or common ratio.

$\sum_{n=1}^{20} 3^{n-2}$	$n=1$	$n=2$	$n=3$
↓	3^{1-2}	3^{2-2}	3^{3-2}
the value we substitute	3^{-1}	3^{0}	3^{1}
to calculate the first term	$\frac{1}{3}$	1	3

By inspection the series is a geometric series. Let us CHECK:

$$r = \frac{T_2}{T_1} = \frac{T_3}{T_2} \quad r = \frac{1}{\frac{1}{3}} = \frac{3}{1} \quad r = 3 = 3 \ , \ r = 3 \ CHECKED! \therefore T_n = a(r)^{n-1} \ , \ T_n = \frac{1}{3}(3)^{n-1}$$

Step:2. Finding the number of terms(n).

$$\sum_{n=1}^{20} 3^{n-2}$$

$$n = (end - start) + 1$$

$$n = (20 - 1) + 1$$

$$n = (19) + 1 \quad \therefore \ n = 20$$

Step:3. Use the correct formula to find the sum.

$$\sum_{n=1}^{20} 3^{n-2} \qquad \frac{1}{3} \qquad 1 \qquad 3$$

$$a = \frac{1}{3} \ r = 3 \ , n = 20 \quad r > 1 \therefore, we\ use: \boldsymbol{S_n = \frac{a(r^n - 1)}{r - 1}}$$

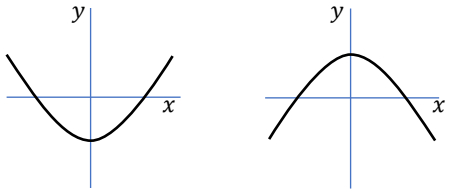
$$S_n = \frac{a(r^n - 1)}{r - 1} \rightarrow S_{20} = \frac{\frac{1}{3}((3)^{20} - 1)}{3 - 1} \rightarrow \boldsymbol{S_{20} = 58\ 113\ 0733.3}$$

FUNCTIONS

1. Parabola
2. Hyperbola
3. Straight-line
4. Exponential
5. Inverses

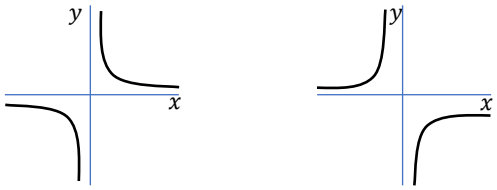
An Overview of the Parabola, Hyperbola, Straight-line, and Exponential function

1. Parabola



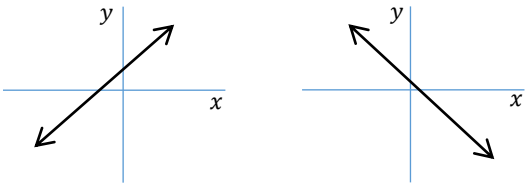
$$f(x) = ax^2 + bx + c$$

2. Hyperbola



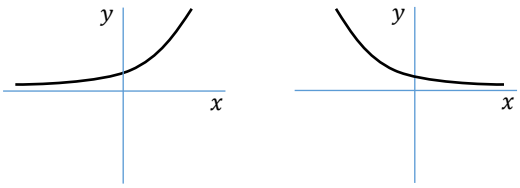
$$f(x) = \frac{a}{x-p} + q$$

3. Straight – Line



$$f(x) = mx + c$$

3. Exponential

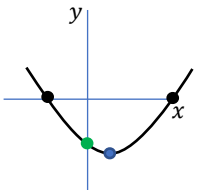


$$f(x) = a.b^{x-p} + q$$

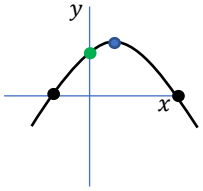
The similarities and difference between the Parabola, Hyperbola, Straight-line, and Exponential function.

1. Parabola

$a > 0$ *or* a *is positive*



$a < 0$ *or* a *is negative*



$f(x) = ax^2 + bx + c$

Requirements to sketch:

a – determines the shape

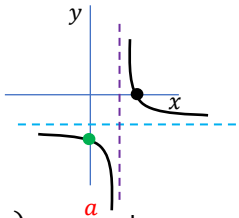
x – intercepts

y – intercept

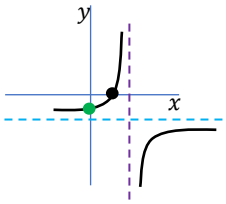
Turning Point.

2. Hyperbola

$a > 0$ *or* a *is positive*



$a < 0$ *or* a *is negative*



$f(x) = \frac{a}{x-p} + q$

Requirements to sketch:

a – determines the shape

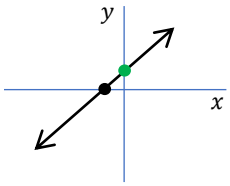
x – intercept

y – intercept

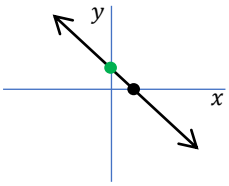
asymptotes: p and q

3. Straight – line

$m > 0$ *or* m *is positive*



$m < 0$ *or* m *is negative*



$f(x) = mx + c$

Requirements to sketch:

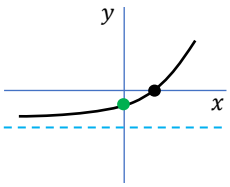
m – determines the shape

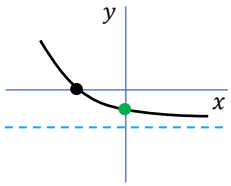
x – intercept

y – intercept

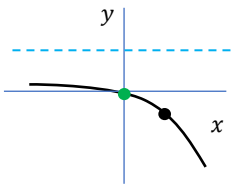
4.Exponential

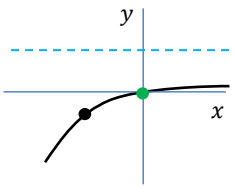
$a > 0$ *or* a *is positive*





$a < 0$ *or* a *is negative*





$f(x) = a.b^{x-p} + q$

Requirements to sketch:

a – determines the shape

x – intercept

y – intercept

asymptote: q

For all functions!

$x-$ intercept: point(s) on the cartesian plane where the graph touches the cartesian plane on the $x-axis$.

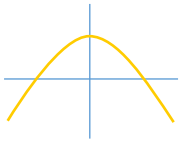
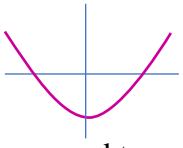
To find the $x-$ intercepts: we make or equate $y=0$, then solve for x. Thus, $(x;0)$.

$y-$ intercept: point on the cartesian plane where the graph touches the cartesian plane on the $y-axis$.

To find the $y-$ intercepts: we makes or equate $x=0$, then solve for y. Thus, $(0;y)$.

1.PARABOLA

The parabola graph is either shaped as

 or 

There are three(3) equations used to represent the parabola:

1. $f(x)=ax^2+bx+c$
2. $f(x)=a(x-p)^2+q$
3. $y=a(x-x_1)(x-x_2)$

NOTE

$x-$ intercept: a point(s) where a graph cuts or touches the $x-axis$.

$y-$ intercept: a point where a graph cuts or touches the $y-axis$.

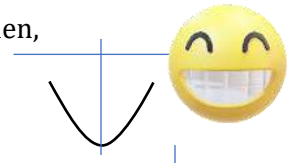
Turning point: a point where the graph changes direction.

The parabola standard format (1): $f(x)=ax^2+bx+c$

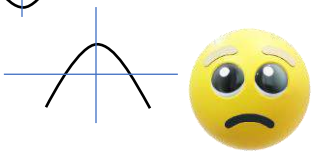
$f(x)=ax^2+bx+c$ → c : $y-$ intercept. The point whereby the graph touches the $y-axis$.

a determines the shape of the graph.

- If a is $+$ (positive) then, the graph is smiling.



- If a is $-$ (negative) then, the graph is sad.



b is used to find the axis of symmetry:

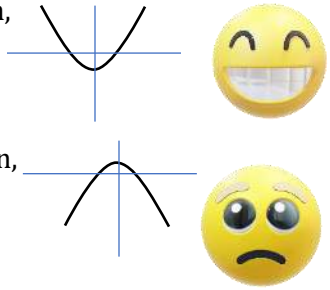
$$x=-\frac{b}{2a}$$

The parabola standard format (2): $f(x) = a(x-p)^2 + q$

$(p\ ;\ q)$ Turning Point

a determines the shape of the graph.

- If a is + (positive) then, the graph is smiling.



- If a is − (negative) then, the graph is sad.

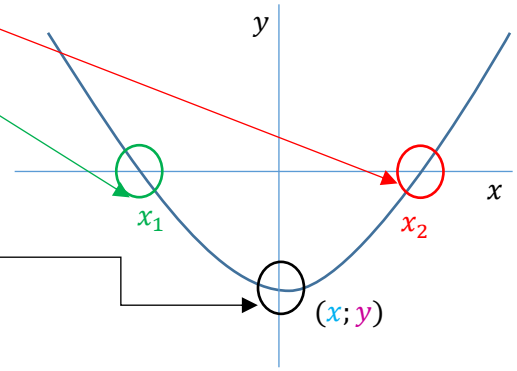
IMPORTANT

Used to find the equation of the parabola when given the **Turning Point**$(p; q)$and **one point** of the graph $(x; y)$.

The parabola standard format (3): $y = a(x-x_1)(x-x_2)$

a determines the shape of the graph, as explained above.

A point on the graph $(x; y)$,



NOTE: The x in $y = a(x-x_1)(x-x_2)$, are identical.

IMPORTANT

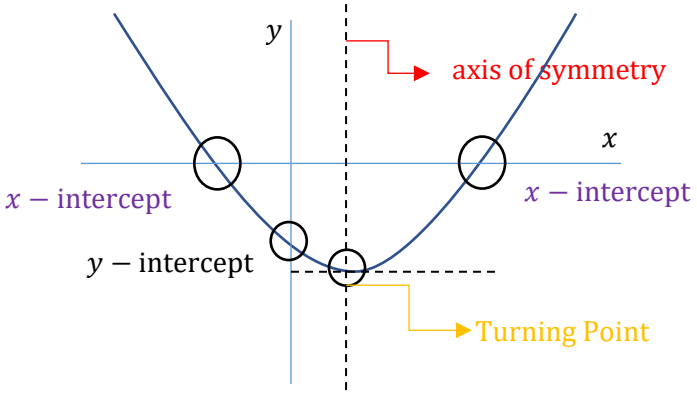
Used to find the equation of the parabola when given **two** $x-$**intercepts**$(x_1$**and** $x_2)$ and **one point** on the graph$(x; y)$.

Converting $f(x) = ax^2 + bx + c$ *to* $f(x) = a(x-p)^2 + q$

Complete the square →

← *Simplify*

1. Graphical concepts and sketching the parabola



$x-$ intercept: point(s) on the cartesian plane where the graph touches the cartesian plane on the $x-axis$.

To find the $x-$ intercepts: we make *or* equate $y=0$, then solve for x. Thus, $(x;0)$.

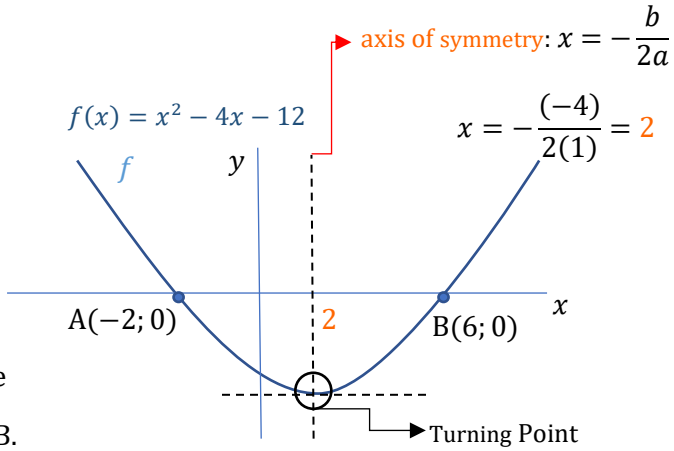
$y-$ intercept: point on the cartesian plane where the graph touches the cartesian plane on the $y-axis$.

To find the $y-$ intercepts: we make *or* equate $x=0$, then solve for y. Thus, $(0;y)$.

Axis of symmetry(A. O. S): a line that divides a graph into two equal halves.

Axis of symmetry: $x=-\frac{b}{2a}$

Alternative methods of finding the axis of symmetry:



1.**Midpoint formula**.

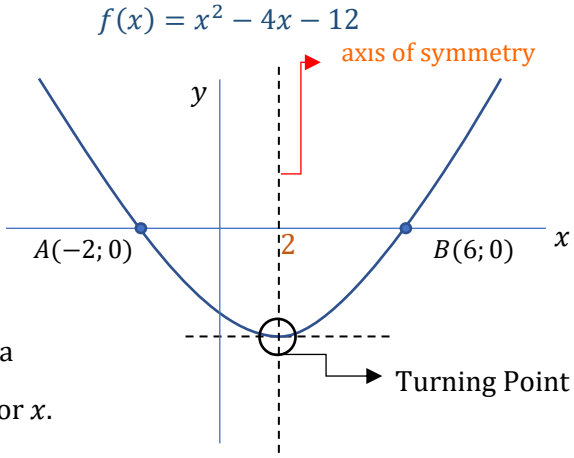
Note that axis of symmetry is the midpoint of A and B. ∴ To find the axis of symmetry we use the midpoint formula because the axis of symmetry halves(is the middle of) AB.

Midpoint of AB: $\left(\frac{x_A+x_B}{2};\frac{y_A+y_B}{2}\right)$, ∴ The axis of symmetry $=\frac{x_A+x_B}{2}$ $\quad$ A. O. S $=\frac{6+(-2)}{2}=2$

2.**Inspection**.

To determine the axis of symmetry we view the middle value(midpoint) between A and B. ∴ A. O. S $=2$

3.**Using the derivative**.



The first derivative of a parabola is a straight − line.

$f(x) = ax^2 + bx + c \quad \rightarrow \quad f'(x) = 2ax + b$

$f'(x) = 2ax + b$, is a straight − line: $y = mx + c$

The axis of symmetry of the parabola is

the x − intercept of the straight − line.

∴ to find the axis of symmetry we derive the parabola

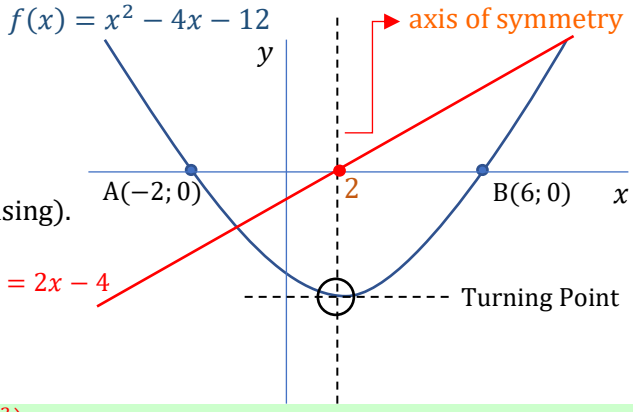
once(first derivative) equate is to ZERO, then solve for x.

Finding $axis\ of\ symmetry$ of the graph of f.

$f(x) = x^2 - 4x - 12$ we derive $f'(x) = 2x^{2-1} - (1).4x^{1-1} - 0 \rightarrow f'(x) = 2x - 4$

$f'(x) = 2x - 4$ we equate to ZERO $2x - 4 = 0 \rightarrow 2x = 4 \rightarrow \frac{2x}{2} = \frac{4}{2} \rightarrow x = 2$

∴ the Axis of symmetry = 2.



Turning Point: the point where the graph changes direction(the graph changes from increasing to decreasing or decreasing to increasing).

Turning Point: $(-\frac{b}{2a}; c - \frac{b^2}{4a})$

Proof of the formula of Turning Point: $\left(-\frac{b}{2a}; c - \frac{b^2}{4a}\right)$ The equation of the parabola: $f(x) = ax^2 + bx + c$

$x - coordinate$ of the turning point(axis of symmetry): $x = -\frac{b}{2a}$, to find the corresponding $y - value\ of$ the turning point we substitute $x = -\frac{b}{2a}$ into $f(x) = ax^2 + bx + c$.

$\therefore f\left(-\frac{b}{2a}\right) = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c \quad \rightarrow \quad f\left(-\frac{b}{2a}\right) = \frac{ab^2}{4a^2} - \frac{b^2}{2a} + c \rightarrow f\left(-\frac{b}{2a}\right) = \frac{2a \times ab^2}{2a \times 4a^2} - \frac{4a^2 \times b^2}{4a^2 \times 2a} + c$

$f\left(-\frac{b}{2a}\right) = \frac{2a^2b^2 - 4a^2b^2}{8a^3} + c \rightarrow f\left(-\frac{b}{2a}\right) = \frac{-2a^2b^2}{8a^3} + c \rightarrow f\left(-\frac{b}{2a}\right) = \frac{-b^2}{4a} + c = c - \frac{b^2}{4a}$

$\therefore$ $the\ y - value$ of the turning point is: $y = c - \frac{b^2}{4a}$ ∴ the Turning Point formula is: $\left(-\frac{b}{2a}; c - \frac{b^2}{4a}\right)$.

SUMMARY OF THE PARABOLA

In this section we are going to reinforce sketching the parabola and finding the equation of the parabola using the 3 methods we covered. Understanding the following summary will increase and solidify your understanding of the parabola.

Given the graph of $f(x) = x^2 - 2x - 8$.

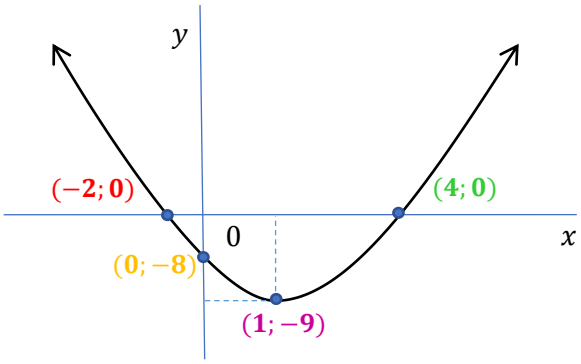
1. Sketch the graph of f, indicate ALL intercepts with the axes and turning point.

2. Use the sketch in question 1 to find the equation of the parabola using: $y = a(x - x_1)(x - x_2)$.

3. Use the sketch in question 1 to find the equation of the parabola using: $y = a(x - p)^2 + q$.

4. A tangent $y = 6x - 24$ touches the graph of $f(x) = ax^2 + bx - 8$ at $(4; 0)$. Determine the equation of f.

1. $x-$ intercept: $x^2 - 2x - 8 = 0 \rightarrow (x - 4)(x + 2) = 0$
$x - 4 = 0 \; or \; x + 2 \rightarrow x = 4 \; or \; x = -2$
$\therefore (-2; 0), (4; 0)$
$y-$ intercept: $y = (0)^2 - 2(0) - 8 \rightarrow y = -8 \;\; \therefore (0; -8)$
Turning point: $\left(-\frac{b}{2a}; c - \frac{b^2}{4a}\right) = \left(-\frac{(-2)}{2.(1)}; -8 - \frac{(-2)^2}{4.(1)}\right)$
$\therefore$ Turning point: $(1; -9)$



2. $(-2; 0)(4; 0)$ $x-$ intercepts
$(0; -8)$ $y-$ intercepts

$y = a(x - x_1)(x - x_2)$
$y = \boldsymbol{a}(x - x_1)(x - x_2)$

$-8 = \boldsymbol{a}(0 - (-2))(0 - (4)) \rightarrow -8 = \boldsymbol{a}(-8)$
$\therefore \boldsymbol{a = 1}$
$\therefore y = \mathbf{1}(x - (-2))(x - (4))$
$y = 1(x + 2)(x - 4)$
$y = 1(x^2 - 4x + 2x - 8)$
$y = 1(x^2 - 2x - 8)$
$y = 1x^2 - 2x - 8$
$\therefore \boldsymbol{f(x) = x^2 - 2x - 8}$

3. $x \;\; y$
$(4; 0) \;\; y = a(x - p)^2 + q \;\; 0 = a(4 - p)^2 + q$

$p \;\; q$
$(1; -9) \;\; 0 = a(4 - p)^2 + q \;\; \rightarrow \;\; 0 = a(4 - 1)^2 - 9$
$0 = a(3)^2 - 9 \rightarrow \;\; a = \frac{9}{9} \;\; \therefore \boldsymbol{a = 1}$

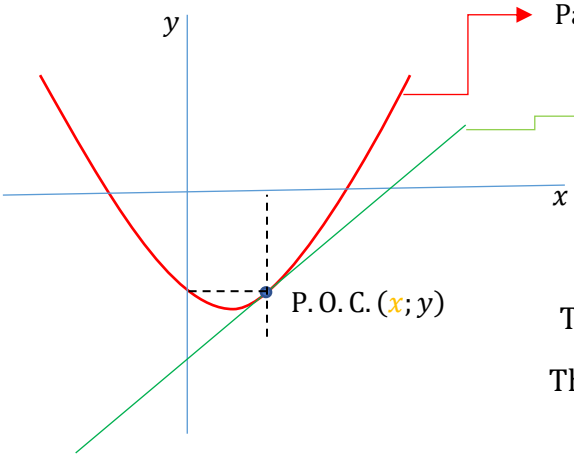
$y = a(x - p)^2 + q \rightarrow \;\; y = 1(x - 1)^2 - 9$

$y = (x - 1)(x - 1) - 9 \rightarrow y = x^2 - x - x + 1 - 9$
$y = x^2 - 2x - 8 \;\;\; \therefore \boldsymbol{f(x) = x^2 - 2x - 8}$

4. Point of Contact $(4; 0)$
$f'(x) = M_{tangent} \;\; g(x) = 6x - 24$
$f(x) = ax^2 + bx - 8$
$f'(x) = 2ax^{2-1} + 1.bx^{1-1} - 0$
$f'(x) = 2ax^1 + bx^0 \rightarrow f'(x) = 2ax + b.1$
$f'(x) = 2ax + b \;\; : 6 = 2ax + b$
$6 = 2a(4) + b \rightarrow \boldsymbol{6 - 8a = b}$
$f(x) = ax^2 + bx - 8 \;\; substitute \; B(4; 0)$
$0 = a(4)^2 + b(4) - 8 \rightarrow \boldsymbol{0 = 16a + 4b - 8}$
$\boldsymbol{6 - 8a = b} \; eq1 \;\; \boldsymbol{0 = 16a + 4b - 8} \; eq2$
$0 = 16a + 4(6 - 8a) - 8 \rightarrow -16a = 16$
$\boldsymbol{a = 1}: 6 - 8a = b \rightarrow 6 - 8(1) = b: \boldsymbol{b = -2}$
$\therefore \boldsymbol{f(x) = x^2 - 2x - 8}$

TANGENTS AND COMPLETING THE SQUARE

Tangent: a straight line that touches the graph at one point. This point is reffered to as the point of contact, (P.O.C).



Parabola: $f(x) = ax^2 + bx + c$

Straight – line: $f(x) = mx + c$

The relationship between the tangent and graph of f.

The first derivative of f = gradient of the straight line

$$\boldsymbol{f'(x) = m(\text{gradient}) \text{ of tangent}}$$

x – coordinate of P. O. C

$$\boldsymbol{f(x) = ax^2 + bx + c}$$

$$f'(x) = 2ax^{2-1} + 1bx^{1-1} + 0$$

$$f'(x) = 2ax + b$$

$$f'(x) = \text{m(gradient) of tangent}$$

$$\boldsymbol{2ax + b = m}$$

x – coordinate of $P.O.C$

Finding the derivative.

a^n *derivative* $n.a^{n-1}$

NOTE

The derivative of a constant is equal to ZERO.

Completing the square

Step:1. The coefficient of x^2 needs to be positive $+1$.

$e.g.$ $x^2 + 6x + 8$

$+1$

Step:2. We take the coefficient of x, which is b. We divide it by 2 and square it: $\left(\frac{b}{2}\right)^2$.

Step:3. Add and substract the solution of $\left(\frac{b}{2}\right)^2$.

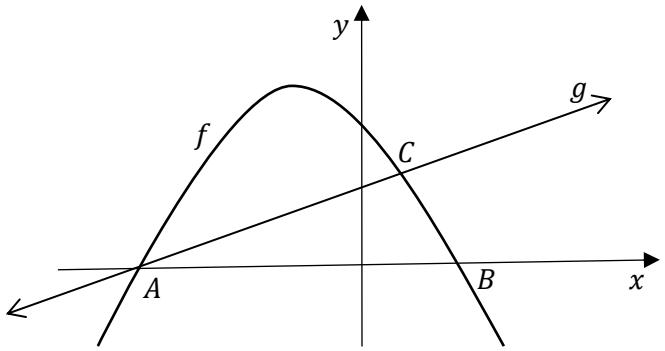
Step:4. Simplify the equation to have two identical brackets.

$e.g.$ $(x + 1)(x + 1) = (x + 1)^2$

CONVERTING $y = ax^2 + bx + c$ TO $y = a(x-p)^2 + q$ BY COMPLETEING THE SQUARE

1.

The figure below represents the sketches of f and g defined by $f(x) = -2x^2 - 4x + 30$ and $g(x) = 2x + 10$. A and B are the $x-$ intercepts of f. The graph of g passes through A. A and C are the points of intersection of f and g. The graph of g intersects the $y-axis$ at D.



1.1. Write down the function of f in the form $f(x) = a(x-p)^2 + q$ and hence, write down the coordinates of the turining points (5)

To convert $f(x) = -2x^2 - 4x + 30$ to the form $f(x) = a(x-p)^2 + q$ we need to complete the square.

Step:1. The coefficient of x^2 needs to be positive $+1$.

$f(x) = -2x^2 - 4x + 30$ the coefficient of f is -2, we need to make $+1$.

$f(x) = -2(x^2 + 2x - 15)$ we took out -2 as a common factor. $f(x) = -2(x^2 + 2x - 15)$.

Note that the coefficient of x^2 *is* $+1$

Step:2. We take the coefficient of x, which is b. We divide it by 2 and square it: $\left(\frac{b}{2}\right)^2$.

$f(x) = -2(x^2 + 2x - 15)$ the coefficient of x *is* $+2 \rightarrow b = 2\left(\frac{b}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = (1)^2 = 1$

Step:3. Add and substract the solution of $\left(\frac{b}{2}\right)^2$.

$f(x) = -2(x^2 + 2x\boxed{\mathbf{+1-1}} - 15)$

Step:4. Simplify the equation to have two identical brackets.

$f(x) = -2(x^2 + 2x + \mathbf{1} - \mathbf{1} - 15) \rightarrow f(x) = -2([x^2 + 2x + 1] - 1 - 15)$

$f(x) = -2\big((x+1)(x+1) - 16\big) \rightarrow f(x) = -2((x+1)^2 - 16) \rightarrow f(x) = -2((x+1)^2 - 16)$

$\boldsymbol{f(x) = -2(x+1)^2 + 32}$ ***or*** $\boldsymbol{f(x) = -2\big(x - (-1)\big)^2 + 32}$ $\therefore$ **Turning Point** $\boldsymbol{(-1; 32)}$.

2.HYPERBOLA

The standard equation of the hyberbola:

$$f(x) = \frac{a}{x-p} + q$$

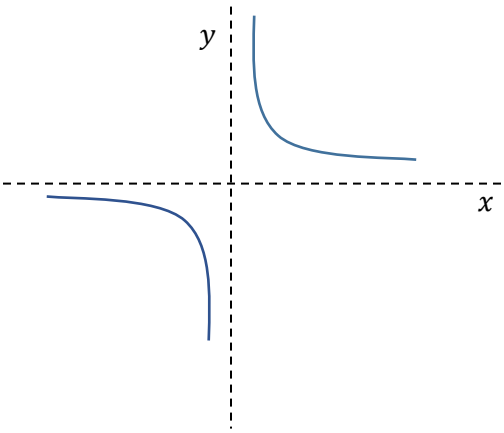
$$f(x) = \frac{a}{x-p} + q$$

a determines the shape of the hyperbola

q → $y-$ horizontal asymptote

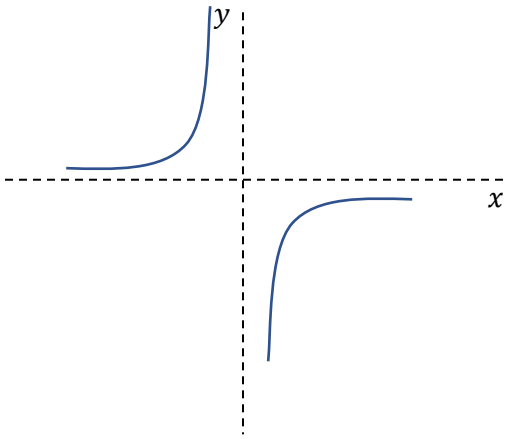
p → $x-$ vertical asymptote

If a is posistive ($a > 0$)



The dotted lines indicate the asymptotes.

If a is negative ($a < 0$)



NOTE

Asymptote(s): lines in which the graph continously approaches but never touches.

Sketching the hyperbola

Step:1. Sketch the asymptotes, both the vertical and horizontal asymptotes.

Step:2. Find the $x-$ intercept, make $y = 0$, solve for x. Find the $y-$ intercept, make $x = 0$, solve for y. Plot the $x-$ intercept and $y-$ intercept.

Step:3. Review the value of $\boldsymbol{a}$(the shape) from $f(x) = \frac{a}{x-p} + q$, if $a > 0$, positive *or* $a < 0$, negative.

Finding the equation of the hyperbola

Step:1. Identify the asymptotes(the vertical and horizontal asymptotes),p and q.

Step:2. Use a point or coordinate on the graph to determine the value of a.

AXIS OF SYMMETRY

The axis of symmetry is the line that divides a graph into two equal halves.

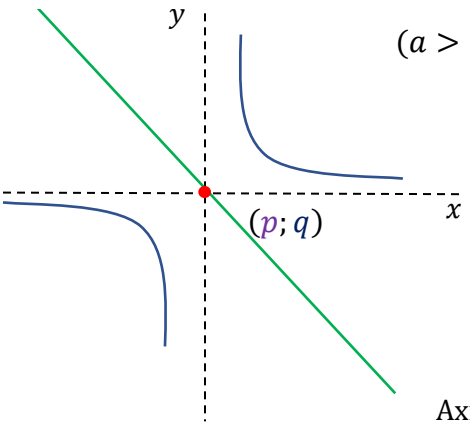
The hyperbola has two(2) axis of symmetry lines.

IMPORTANT

The axis of symmetry lines pass the point of intersection of the asymptotes$(p; q)$.

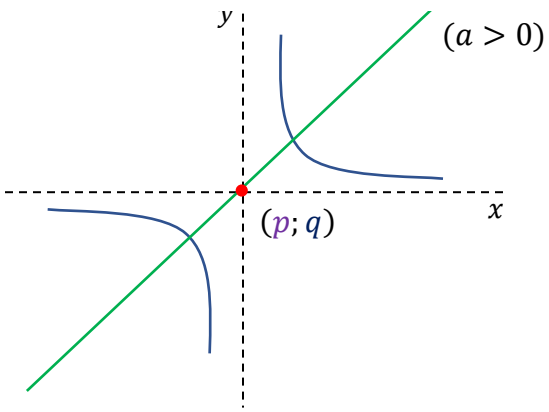
The axis of symmetry with a **negative gradient**:

$y = -x + c$ *or* $y = -x + p + q$



The axis of symmetry with a **positive gradient**:

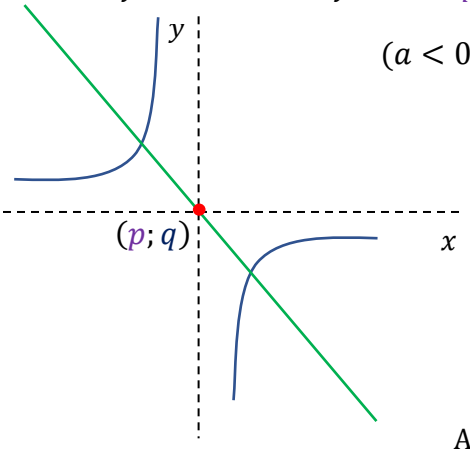
$y = x + c$ *or* $y = x - p + q$



Axis of symmetry lines

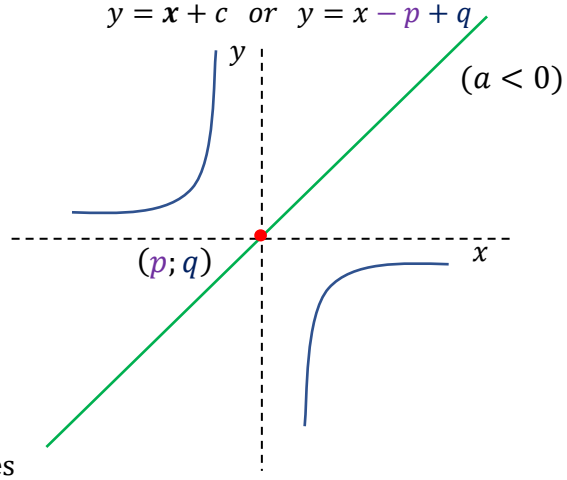
The axis of symmetry with a **negative gradient**:

$y = -x + c$ *or* $y = -x + p + q$



The axis of symmetry with a **positive gradient**:

$y = x + c$ *or* $y = x - p + q$

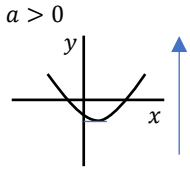
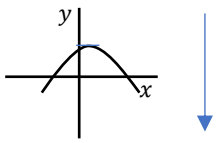
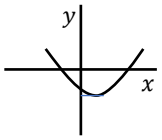
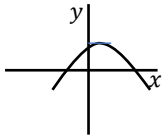
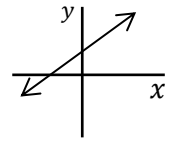
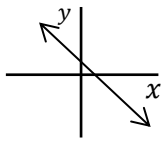
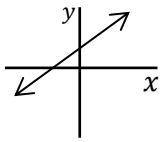
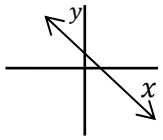
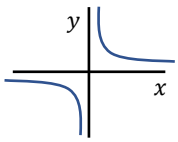
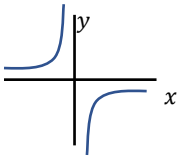
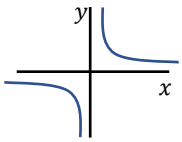
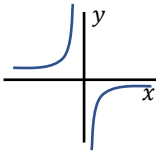
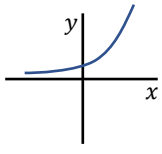
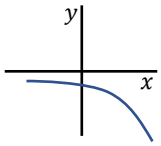
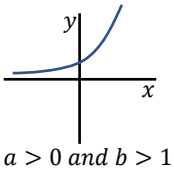
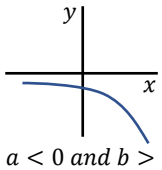


Axis of symmetry lines

RANGE AND DOMAIN OF FUNCTIONS

Range: a set of $y-values$ that a function is valid.

Domian: a set of $x-values$ that a function is valid.

Range	Domain
1.Parabola: $\boldsymbol{f(x)=ax^2+bx+c}$ $a>0$  $a<0$  **Range:** $\boldsymbol{y \geq c-\frac{b^2}{4a}}$ **Range:** $\boldsymbol{y \leq c-\frac{b^2}{4a}}$	Parabola: $f(x)=ax^2+bx+c$ $a>0$  $a<0$  Doman: $x \in \mathbb{R}$ Domain: $x \in \mathbb{R}$
2.Straight line: $\boldsymbol{f(x)=mx+c}$ $m>0$  $m<0$  **Range:y**$\in \mathbb{R}$ **Range:y**$\in \mathbb{R}$	Straight line: $f(x)=mx+c$ $m>0$  $m<0$  Domain: $x \in \mathbb{R}$ Domain: $x \in \mathbb{R}$
3.Hyperbola: $\boldsymbol{f(x)=\frac{a}{x-p}+q}$ $a>0$  $a<0$  **Range:** $\boldsymbol{y \in \mathbb{R}, y \neq q}$ **Range:** $\boldsymbol{y \in \mathbb{R}, y \neq q}$	Hyperbola: $:f(x)=\frac{a}{x-p}+q$ $a>0$  $a<0$  Domain: $x \in \mathbb{R}, x \neq p$ Domain: $x \in \mathbb{R}, x \neq p$
4.Exponential: $\boldsymbol{f(x)=a.b^{x-p}+q}$  $a>0$ *and* $b>1$  $a<0$ *and* $b>1$ **Range:** $\boldsymbol{y>q}$ **Range:** $\boldsymbol{y<q}$	Exponential: $f(x)=a.b^{x-p}+q$  $a>0$ *and* $b>1$  $a<0$ *and* $b>1$ Domain: $x \in \mathbb{R}$ Domain: $x \in \mathbb{R}$

INVERSES

Inverses are functions that are reflected about the line $y = x$.

1. Exponential function	2. Parabola	3. Straight line
$f(x) = a^x$	$f(x) = ax^2$	$f(x) = mx + c$

Steps to finding inverses:

Step:1. Swap-Interchange x and y.

Step:2. Make y the subject of the formula.

Inverse Notation.

$f^{-1}(x)$ – indicates the inverse of a one – to – one function.

Reflection about the $y = x$ is the same as the inverse.

Range and Domain of Inverses.

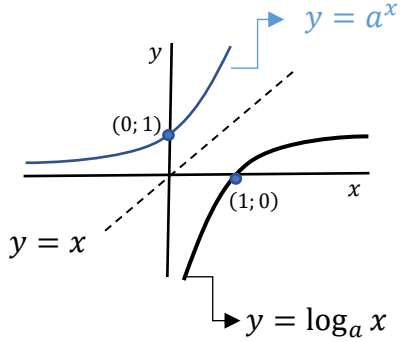
Function:	Inverse:
Domain	Domain
Range	Range

1. $y = a^x$

To find the inverse:

Step:1. $x = a^y \rightarrow$ *Use logs: number = base^{power}* $\rightarrow \log_{base} number = power$

Step:2. $\log_a x = y \rightarrow y = \log_a x$



Range and Domain:

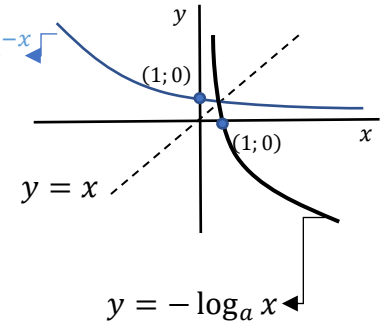
$y = a^x$	$y = \log_a x$
D: $x \epsilon \mathbb{R}$	D: $x > 0$
R: $y > 0$	R: $y \epsilon \mathbb{R}$

$y = a^{-x}$

To find the inverse:

Step:1. $x = a^{-y} \rightarrow$ *Use logs: number = base^{-power}* $\rightarrow \log_{base} number = -power$

Step:2. $\log_a x = -y \rightarrow y = -\log_a x$



Range and Domain:

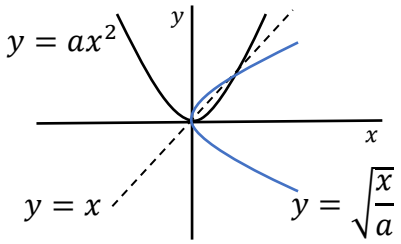
$y = a^{-x}$	$y = -\log_a x$
D: $x \epsilon \mathbb{R}$	D: $x > 0$
R: $y > 0$	R: $y \epsilon \mathbb{R}$

2. $y = ax^2$

To find the inverse:

Step:1. $x = ay^2$

Step:2. $\frac{x}{a} = y^2 \rightarrow y = \sqrt{\frac{x}{a}}$

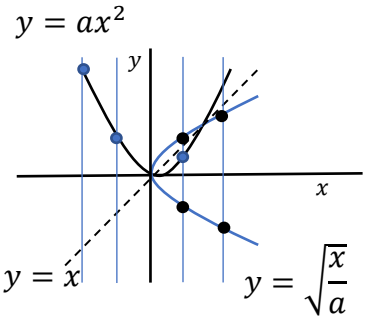


NOTE: $y = \sqrt{\frac{x}{a}}$ is not a function. For the inverse to be a function, every $x-$value should have 1 corresponding $y-$value, this is called a **one − to − one function**.

Therefore, we need to restrict $y = \sqrt{\frac{x}{a}}$ $(x \geq 0 \text{ or } x \leq 0)$

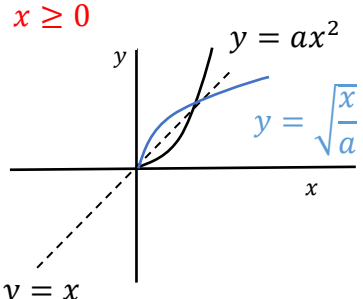
To check if a graph is a function it needs to pass a **vertical line test**. Where vertical lines are drawn on the graph and should only **touch or intersect** the graph **at one point only** if the graph is a **function**.

Vertical Line Test

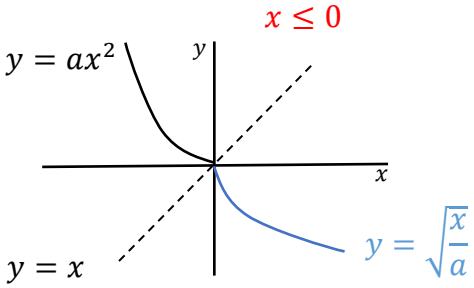


$y = ax^2$ passes the line test, the lines intersect the graph at one vertical point only, therefore, $y = ax^2$ is a function. $y = \sqrt{\frac{x}{a}}$ fails the line test, the vertical line intersects the graph twice, therefore it is not a function.

$x \geq 0$



$x \leq 0$



$y = \sqrt{\frac{x}{a}}$ passes the line − test ∴ it is a **function**.

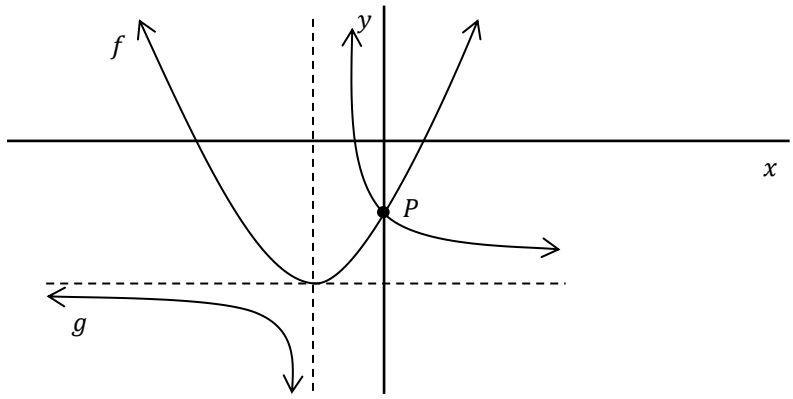
A fish

A bird

EXAM CALCULATIONS ON FUNCTIONS

1.

The graph of $f(x) = a(x+p)^2 + q$ and $g(x) = \frac{2}{x+1} - 3$ are sketched below. P is the $y-$ intercept of f and g. The horizontal asymptote of g is also a tangent to f at the turning point of f.



1.1. Write down the equation of the vertical asymptote of g. (1)

1.2. Determine the coordinates of P. (2)

1.3. Determine the equation of f. (3)

1.4. One of the axis of symmetry of g is a decreasing function. Write down the equation of this axis of symmetry. $h(x)$. (2)

1.5. For which values of k *will* $g(x) = h(x) + k$ have TWO real roots that are of opposite signs? (2)

1.6. Give the domain of $m(x)$ *if* $m(x) = g(2x) + 5$ (3)

Interpretation:

$f(x) = a(x+p)^2$ $g(x) = \frac{2}{x+1} - 3$	In the diagram the turning point of f is at the
T. P$(-p; q)$ Asymptotes: $x = -1$ *or* $p = -1$	intersection of the asymptotes of g, $(-1; -3)$.
$y = -3$ *or* $q = -3$	f and g share the same $y-$ intercept P.

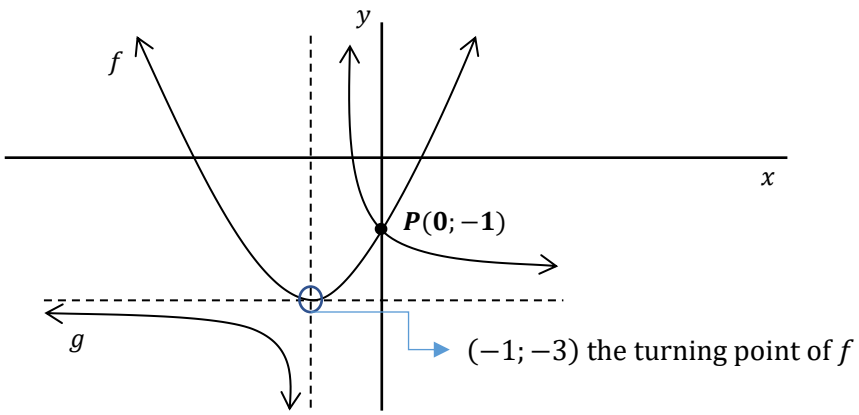
1.1. $g(x) = \dfrac{2}{x+1} - 3$ the vertical asymptote of g: $x = -1$.

1.2. P is the $y-$ intercept of g. $y-$ intercept, make $x = 0$, solve for y

$$g(x) = \frac{2}{x+1} - 3 \quad \rightarrow \quad y = \frac{2}{x+1} - 3 \quad \rightarrow \quad y = \frac{2}{0+1} - 3 \quad \rightarrow \quad y = \frac{2}{1} - 3 \quad \rightarrow \quad y = -1$$

$\therefore \boldsymbol{P(0;-1)}$

1.3.



We have the turning point $(\overset{p}{-1}; \overset{q}{-3})$ and a point on the graph $P(\overset{x}{0}; \overset{y}{-1})$: We use: $f(x) = a(x-p)^2 + q$

Substitute p and q: $y = a\big(x-(-1)\big)^2 - 3 \quad \rightarrow \quad y = a(x+1)^2 - 3$

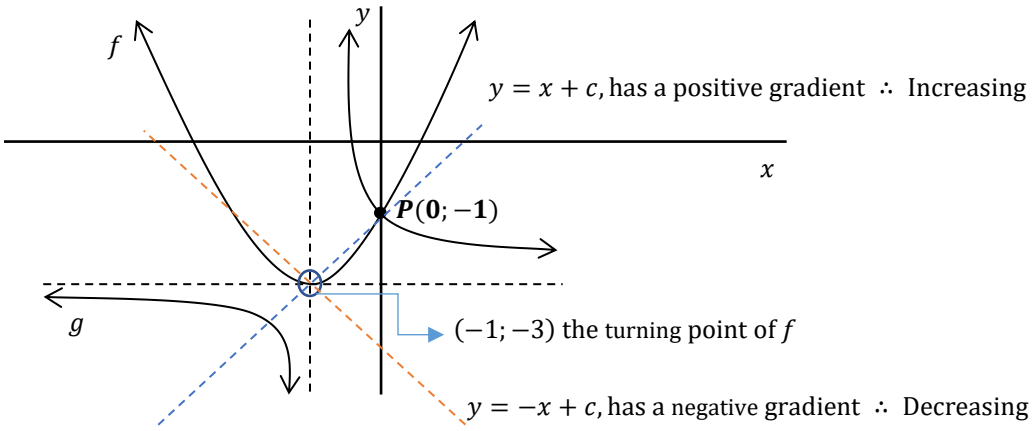
Substitute x and y: $-1 = a(0+1)^2 - 3 \quad \rightarrow \quad -1 = a(1)^2 - 3 \quad \rightarrow \quad -1 + 3 = a \quad \therefore \boldsymbol{a = 2}$

We have: $\boldsymbol{a = 2, p = -1}$ **, and** $\boldsymbol{q = -3}$ $\therefore$ **substitute into** $\boldsymbol{f(x) = a(x-p)^2 + q}$

$\boldsymbol{f(x) = 2\big(x-(-1)\big)^2 - 3}$ $\therefore$ **the equation of** $\boldsymbol{f}$ **is:** $\boldsymbol{f(x) = 2(x+1)^2 - 3}$.

1.4.

The graph of f has two axis of symmetry lines.



The decreasing function has a negative gradient.

$y = -x + c$ passes $(-1; -3)$. We substitute this point and *solve for c*.

$-3 = -(-1) + c \quad \rightarrow \quad -3 = 1 + c \quad \rightarrow \quad -3 - 1 = c \quad \rightarrow \quad c = -4.$

$\therefore \boldsymbol{h(x) = -x - 4}$ is the equation of the axis of symmetry of a decreasing function.

1.5.

$y = -x - 4$

For $g(x) = h(x) + k \quad \rightarrow \quad g(x) = -x - 4 + k$ to have TWO real roots of thats are of opposite signs k should be: $k > 3$.

1.6.

$$m(x) = g(2x) + 5 \quad g(x) = \frac{2}{x+1} - 3 \ : \ g(2x) = \frac{2}{2x+1} - 3 \rightarrow g(2x) = \frac{2}{2\left(\frac{2x}{2} + \frac{1}{2}\right)} - 3$$

$$g(2x) = \frac{2}{2\left(x + \frac{1}{2}\right)} - 3 \rightarrow g(2x) = \frac{1}{\left(x + \frac{1}{2}\right)} - 3 \qquad m(x) = g(2x) + 5 \rightarrow \quad m(x) = \frac{1}{\left(x + \frac{1}{2}\right)} - 3 + 5$$

$$\boldsymbol{m(x) = \frac{1}{\left(x + \frac{1}{2}\right)} + 2 \ , p = -\frac{1}{2}} \qquad \therefore \textbf{Domain}: \ x \in \mathbb{R}, x \neq -\frac{1}{2}$$

2.RULES OF DIFFERENTIATION

Finding the derivative USING RULES OF DIFFERENTIATION.

Rule:

NOTE

$f'(x)$ indicates the derivative of a function and f^{-1} indicates the inverse of a function.

1. $\boldsymbol{f(x) = a}$ **(** $\boldsymbol{a}$ **is contant − any the number)**

$f'(x) = 0$

The derivative of a constant is equal to zero(0).

Example:

1. $f(x) = -4$ $\quad f'(x) = 0$

2. $f(x) = 56$ $\quad f'(x) = 0$

3. $f(x) = 1507$ $\quad f'(x) = 0$

2. $\boldsymbol{f(x) = x^n}$ **The Power Rule**

$f'(x) = n.x^{n-1}$

The derivative of an exponential is the exponent multiplied by the expression, then subtract one(1) from the exponent.

Example: **1**.

a. $f(x) = -2x^2 + 1$

$f'(x) = 2 \times (-2x^{2-1}) + 0$

$\therefore \boldsymbol{f'(x) = -4x}$

b. $f(x) = 2x^2 - 7x + 6$

$f'(x) = 2 \times (2x^{2-1}) - 1 \times 7x^{1-1} + 0$

$f'(x) = 4x - 7x^0 \quad \therefore f'(x) = 4x - 7(1) \quad \therefore \boldsymbol{f'(x) = 4x - 7}$

Notation.

There are four notaions used to represent the derivative of a function.

We will use $f(x) = 7x^2 + 3x$ as a function and as our example.

- Function: $f(x) = ..$ $\quad f(x) = 7x^2 + 3x \rightarrow$ Derivative $f'(x) = 14x + 3$
- Function: $y = ..$ $\quad y = 7x^2 + 3x \rightarrow$ Derivative $\frac{d_y}{d_x} = 14x + 3$
- Function: $D_x[function]$ $\quad D_x[7x^2 + 3x] \rightarrow$ Derivative $D_x[7x^2 + 3x] = 14x + 3$
- Function: $\frac{d}{d_x}[function]$ $\quad \frac{d}{d_x}[7x^2 + 3x] \rightarrow$ Derivative $\frac{d}{d_x}[7x^2 + 3x] = 14x + 3$

Important Conditions:

1. Always make your expression linear, by eliminating fractions.

Example:

Derive with respect to x: $y = \frac{3}{x} + x^{\frac{1}{3}}$

$\therefore y = 3x^{-1} + x^{\frac{1}{3}} \rightarrow \frac{d_y}{d_x} = -1.3x^{-1-1} + \frac{1}{3}.x^{\frac{1}{3}-1} \rightarrow \frac{d_y}{d_x} = -3x^{-2} + \frac{1}{3}x^{-\frac{2}{3}} \rightarrow \therefore \boldsymbol{\frac{d_y}{d_x} = -3x^{-2} + \frac{1}{3\sqrt[3]{x^2}}}$

2. Eliminate surds – (square roots).

Example:

Determine the derivative of: $f(x) = 2x^2 + \sqrt{x^3}$

$\therefore f(x) = 2x^2 + x^{\frac{3}{2}} \rightarrow f'(x) = 2.2x^{2-1} + \frac{3}{2}.x^{\frac{3}{2}-1} \rightarrow f'(x) = 4x + \frac{3}{2}x^{\frac{1}{2}} \rightarrow \boldsymbol{f'(x) = 4x + \frac{3\sqrt{x}}{2}}$

3. Eliminate brackets.

Example:

Evaluate: $D_x[(x^3 - 3)^2]$

$\therefore D_x[(x^3 - 3)(x^3 - 3)] \rightarrow D_x[x^6 - 3x^3 - 3x^3 + 9] \rightarrow D_x[x^6 - 6x^3 + 9]$ *we can derive,*

$D_x[x^6 - 6x^3 + 9] = \boldsymbol{6x^5 - 18x^2}$

4. Ensure that y is subject of the equation.

Example:

Determine the derivative of: $2xy = 2x^2 - 7x + 6$

$\therefore$ we make y subject of the equation: $\frac{2xy}{2x} = \frac{2x^2}{2x} - \frac{7x}{2x} + \frac{6}{2x} \quad y = x + \frac{7}{2} + \frac{2}{x} \rightarrow y = x + \frac{7}{2} + 2x^{-1}$

$\frac{d_y}{d_x} = 1 + 0 + (-1).2x^{-1-1} \rightarrow \frac{d_y}{d_x} = 1 - 2x^{-2} \rightarrow \boldsymbol{\frac{d_y}{d_x} = 1 - \frac{2}{x^2}}$

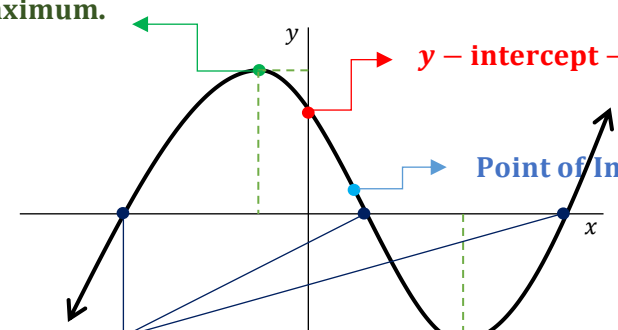
CUBIC FUNCTION

CUBIC FUNCTION OUTLINE

Standard form: $f(x) = ax^3 + bx^2 + cx + d$

Turning Point – local maximum.

$f'(x) = 0$



y – intercept – (d)

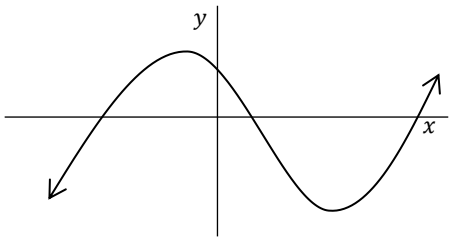
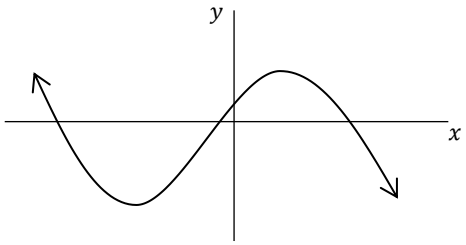
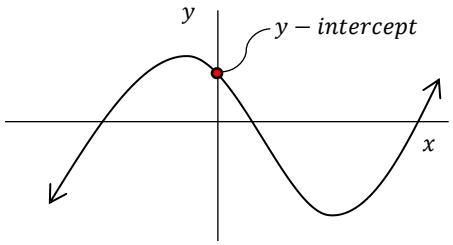
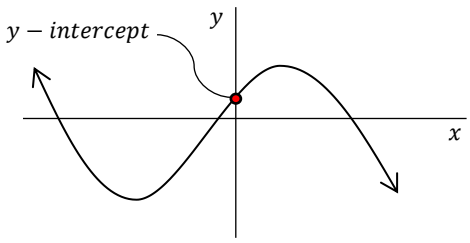
Point of Inflection – $f''(x) = 0$

x – intercepts – $f(x) = 0$

Turning Point – local minimum.

$f'(x) = 0$

$f(x) = ax^3 + bx^2 + cx + d$

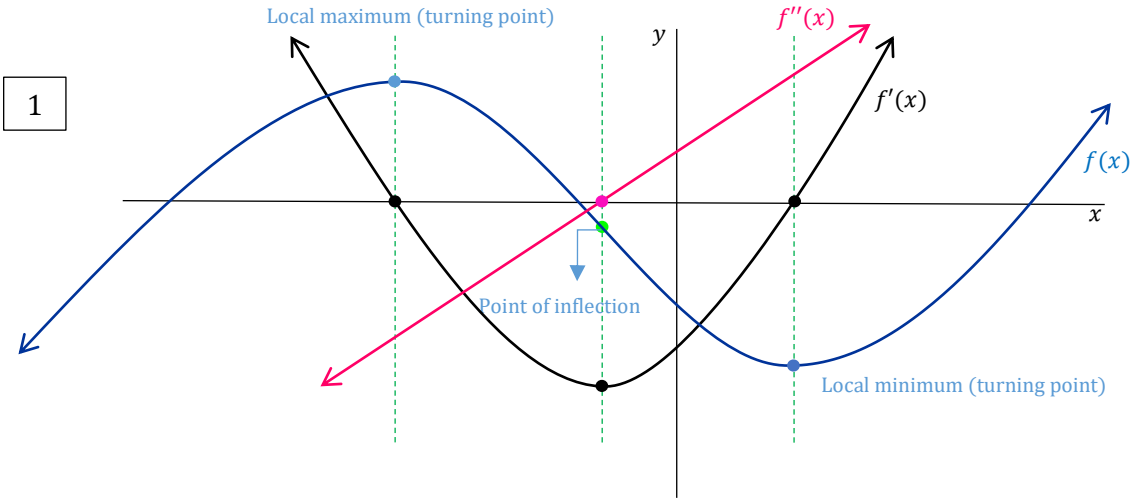
The role of a. a is the shape of the graph. If a is positive *or* $a > 0$ 	If a is negative *or* $a < 0$ 
The role of d. d is the y – intercept of the graph. 	

6.THE RELATIONSHIP BETWEEN THE CUBIC, QUADRATIC AND STRAIGHT-LINE GRAPH

Cubic graph $f(x) = ax^3 + bx^2 + xc + d$ Quadratic graph(Parabola) $\boldsymbol{f'(x) = ax^2 + bx + c}$

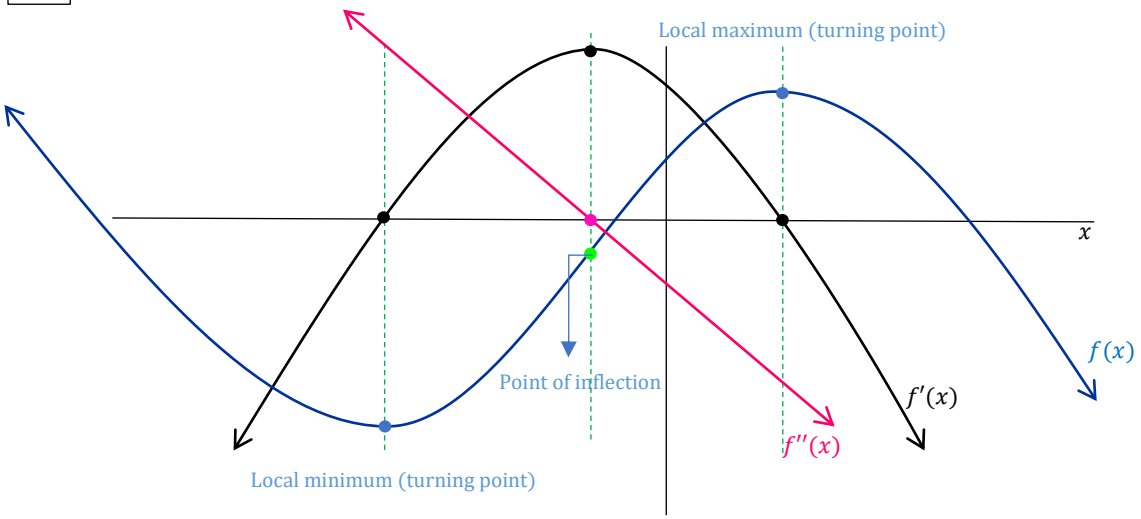
Straight – line graph $f''(x) = mx + c$

1



AND

2

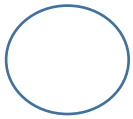
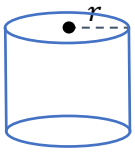
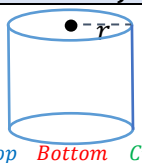
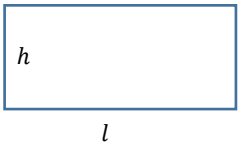
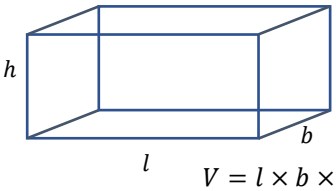
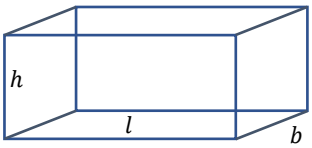
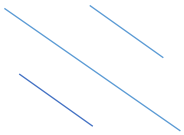
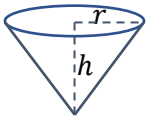
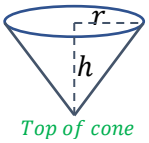
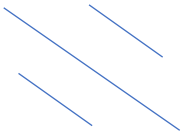
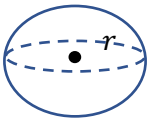
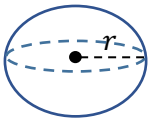
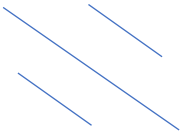
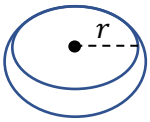
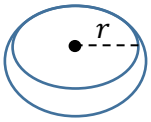
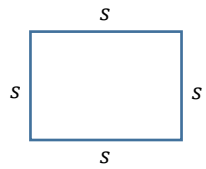
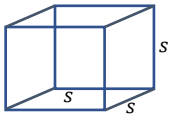
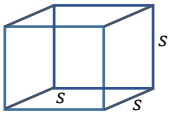


Tip!

When answering questions on cubic graphs, be able to draw all three graphs based on the question given, this will increase your understanding of the question.

APPLICATION OF CALCULUS

Fundamental formulas to know.

Area $unit^2$	**Volume** $units^3$	**Surface Area** $units^2$
Circle  $A = \pi r^2$	*Cylinder*  $V = \pi r^2 . h$	 *Top* *Bottom* *Curved surface area of cylinder* $Surface\ Area = (\pi r^2 + \pi r^2) + 2\pi rh$ $Surface\ Area = 2\pi r^2 + 2\pi rh$
Rectangle  $A = l \times h$	 $V = l \times b \times h$	 *Top + Bottom* *Front + Back* *Left + Right* $S.Area = 2(l \times b) + 2(l \times h) + 2(b \times h)$
Cone 	 $V = \frac{1}{3}\pi r^2 . h$	 *Top of cone* *Curved outer area of cone* $Surface\ Area = \pi r^2 + \pi rh$
Sphere 	 $V = \frac{4}{3}\pi r^3$	 $Surface\ Area = 4\pi r^2$
Hemisphere 	 $V = \frac{2}{3}\pi r^3$	 $Surface\ Area = \left(\frac{4\pi r^2}{2}\right) + \pi r^2$ *Top of hemisphere* $= (2\pi r^2) + \pi r^2 = 3\pi r^2$ *Curved outer area of the heminsphere*
Square  $A = s \times s$	 $V = s \times s \times s$	 *Top + Bottom* *Front + Back* *Left + Right* $S.Area = 2(s \times s) + 2(s \times s) + 2(s \times s)$

FINANCIAL MATH REVISION

Simple Interest and Simple Decay:

- $A = P(1 + i.n) \rightarrow$ Growth
  $\rightarrow$ Hire Purchase
- $A = P(1 - i.n) \rightarrow$ Simple Decay

Principal amount — Interest Rate

$$A = P(1 \pm i.n)$$

Number of years(period)

Final amount(money accumulated)

Compound Interest and Reducing Balance:

- $A = P(1 + i)^n \rightarrow$ Growth
  $\rightarrow$ Inflation
- $A = P(1 - I)^n \rightarrow$ Depreciation
  $\rightarrow$ Reducing Balance

Principal amount — Interest Rate(divided by the months)

$$A = P(1 \pm i)^n$$

Number of years(period) multiplied by months.

Final amount(money accumulated)

Nominal and Effective interest rates:

$$1 + i^{effective} = \left(1 + \frac{i^{nominal}}{k}\right)^k$$

Used to calculate effective interest or to calculate nominal interest rate.

k: is the number of times per year the interest is calculated.

$i^{nominal}$: is the qouted interest rate.

$i^{effective}$: is the actual interest rate received after compounding has been considered(i. e. annualy, monthly, semi − annually or quaterly).

If you are qouted 10% by a bank for a loan the resulting effective interest rate will be diffeent depending on weather the loan will be compounded annually, monthly, semi − annually, or quaterly.

Example:

Andani has just made R250 000 from his farming business, he wants to invest this money and he is faced with two options. Bank A offers him 10% p. a. compounded monthly and Bank B offers him 10%p. a. compounded semi − annually. Which bank is offering a good deal?

$$Bank\ A: 1 + i^{effective} = \left(1 + \frac{i^{nominal}}{k}\right)^k = 1 + i^{effective} = \left(1 + \frac{0{,}1}{12}\right)^{12} = i^{effective} = 0{,}1047 \times 100 = 10{,}47\%$$

$$Bank\ B: 1 + i^{effective} = \left(1 + \frac{i^{nominal}}{k}\right)^k = 1 + i^{effective} = \left(1 + \frac{0{,}1}{2}\right)^{2} = i^{effective} = 0{,}1025 \times 100 = 10{,}25\%$$

∴ **Bank A is offering a good deal!**

ANNUITITES

Annuities are a number of **fixed** amounts of **monthly instalments** or **payments** at a given interest rate(i), over a given time(n). There are two types of annuities, Future Value and Present Value.

Future Value – includes fixed(equal) monthly instalments, payments or deposits into **investment or savings account** to provide an accumulated amount after a given period(n), at a given compound interest at a certain interest rate(i).

$$\textbf{Future Value} = \frac{x[(1+i)^n - 1]}{i}$$

i – *interest rate divided by months*, n – years × months

x – fixed(equal)monthly payments or deposits

- Savings
- Pension(annuities)
- Funeral
- Sinking Fund Value

Key words

REMEMBER

In **Future Value Anuitties** payments are made at the end of the period(month) over a given time period(n).

Tip!

In Future Value annuitties, payments are made at the end of the period. When payments are made at the beginning of the month we use the formula: $FV = \frac{x[(1+i)^n - 1]}{i} \times (1+i)$

Present Value – includes fixed(equal)monthly instalments, payments or deposits to pay back **a loan or bond** over a given period(n) , (reducing balance on loan)at a given compound interest at a certain interest rate(i). The money is recieved before it is paid back.

$$\textbf{Present Value} = \frac{x[1-(1+i)^{-n}]}{i}$$

i – interest rate divided by months, n – years × months

x – fixed(equal)monthly payments or deposits

- Loans
- Borrow
- Bonds(Houses)
- Retirement(money recieved)

Key words

REMEMBER

In **Present Value Anuitties** payments are made 1 period(month) after being granted.

OUTSTANDING BALANCE

The amount of money due to finish(settle) a loan.

Outstanding Balance = Loan(amount of the loan + interest the laon accrued) – Repayments(over n years including interest)

Outstanding Balance = Accumulated value of loan – Future Value

Outstanding Balance $= \boldsymbol{P(1+i)^n - \frac{x[(1+i)^n - 1]}{i}}$

SINKING FUND

A savings account set up to replace the current item in the future.

Companies often purchase large and expensive equiptment to use for a certain time period. When this time period is over, new equipment will be required to replace to old equpment. The old equipment would have depreciated in value(**scrap value**: $\mathbf{A = P(1-i)^n}$) and the new equipment would be more expensive than the intial purchase price of the old equipment, this is due to inflation, therefore, the **new equipment** value will be: $\boldsymbol{A = P(1+i)^n}$.

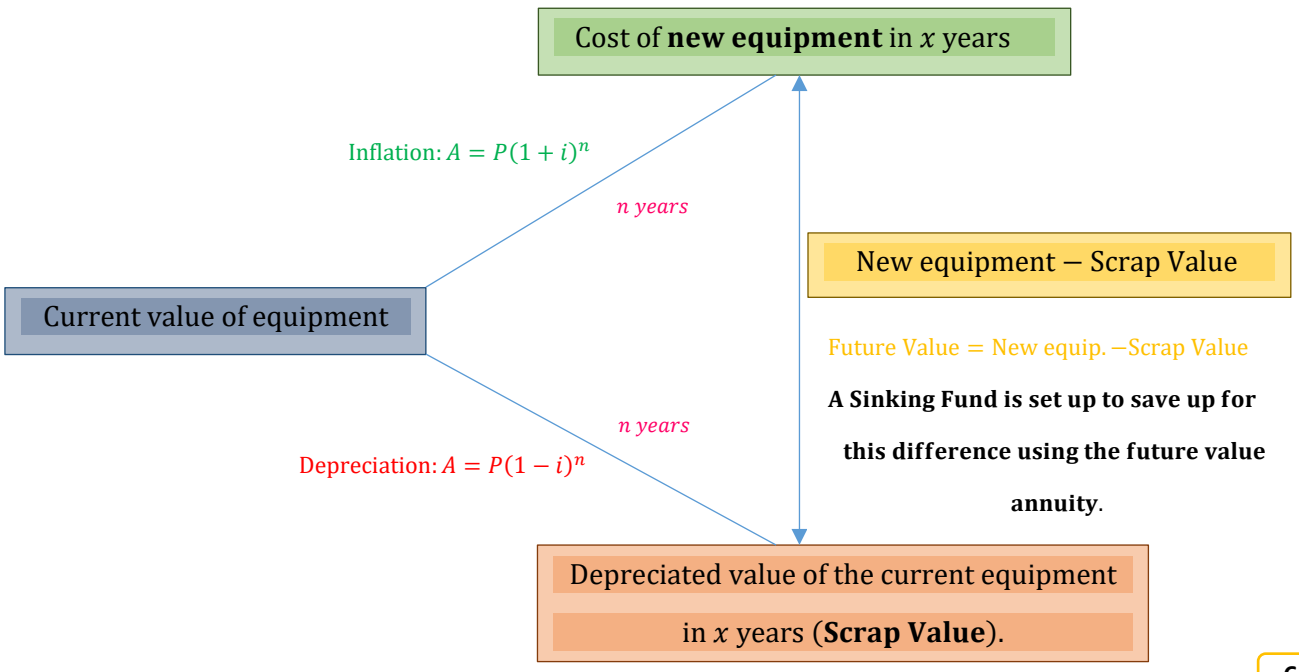
The companies have to pay the difference between the cost of the **new equipment** − **the scrap value**.

Companies are smart and invest money monthly in advance to pay thid difference, this fund is called a **sinking fund**. The monthy investment are for the future, hence, Future Value is used:

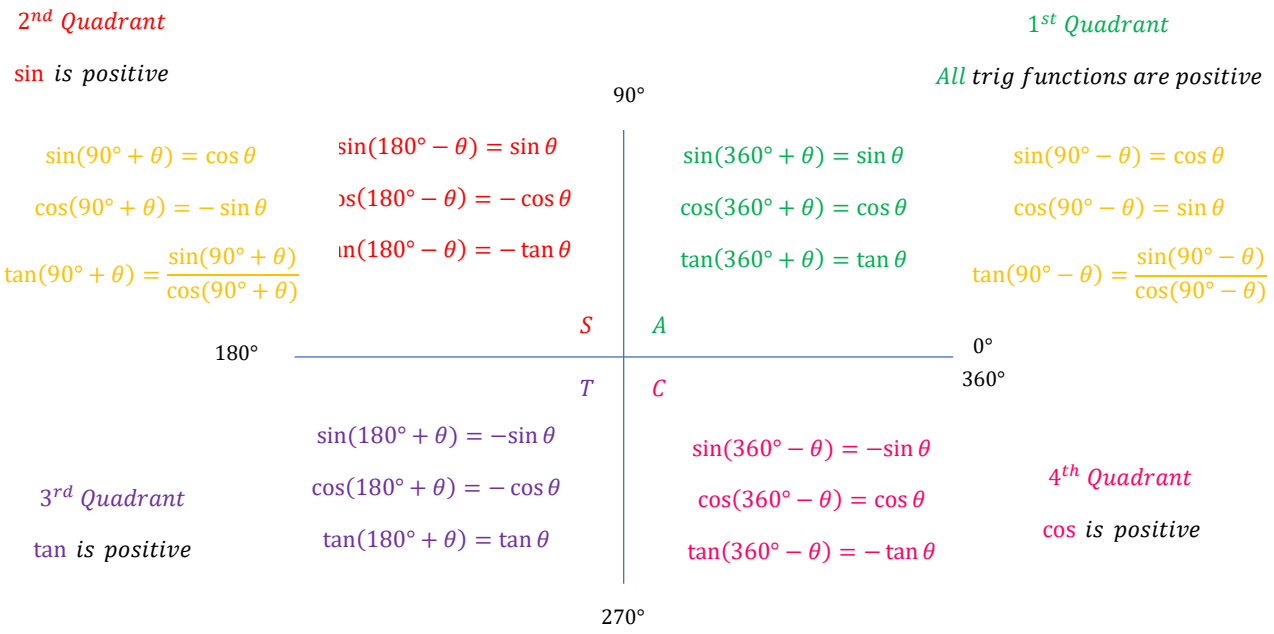
Future Value $= \boldsymbol{\frac{x[(1+i)^n - 1]}{i}}$

Sinking Fund Value = The cost of new equipment − The scrap value

(inflated equipment in n years) (depreciation equipment in n years)



REDUCTION FORMULAE



Applying the Reduction Formulae

Important: $e.g. \sin(360° + \theta)$

$\sin(360° + \theta)$

θ is the angle we reduce in terms of.

$360° +$ tells you the quadrant. Which is Quadrant 1.

When reducing ALWAYS! ask yourself these questions:

Step1: Which trigonometric function am I am reducing? (sin, cos, or tan).

Step2: Which Quadrant is the expression in?

$e.g. \cos(180 - \theta)$

$180° -$ *is the* 2^{nd} *Quadrant*

Step3: What sign is trigonometric function in that Quadrant?

$e.g. \cos(180 - \theta)$

$\cos$ *is negative* $(-)$ *in the* 2^{nd} *Quadrant*, $\therefore \cos(180 - \theta) = -\cos\theta$

ANALYTICAL GEOMETRY

ANALYTICAL GROMETRY REVISION

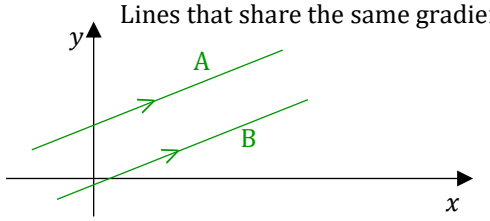
Straight – Line:

$y = mx + c$

$$\text{gradient(m)} = \frac{y_2 - y_1}{x_2 - x_1} \qquad c = y - intercept$$

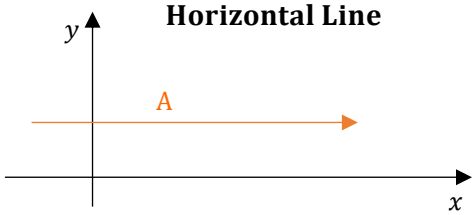
Parallel Lines ∥:

Lines that share the same gradient.



The gradient of line A is equal to the gradient of line B. $M_A = M_B$

Horizontal Line

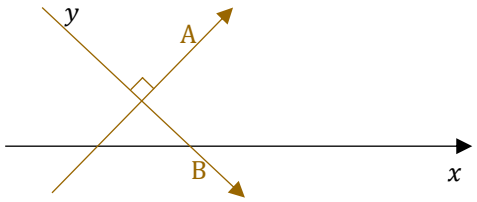


The gradient of a horizontal line is ZERO.

$M_A = 0$

Perpendicular Lines ⊥:

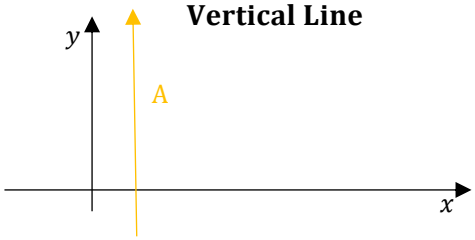
Lines that meet at a 90°angle.



The gradient of line A multplied by the gradient of line B is equal to negative one(-1).

$M_A \times M_B = -1$

Vertical Line

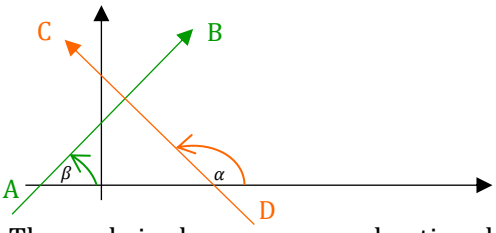


The gradient of a horizontal line is UNDEFINED.

M_A = UNDEFINED

Angle of Inclination:

The angle of inclination is the angle the straight – line makes with the $x - axis$.

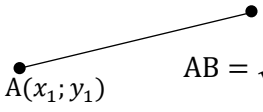


The angle is always measured anti – clockwise.

$\tan\beta = \text{gradient}_{AB}$ $\tan\alpha = \text{gradient}_{CD}$

Distance formula:

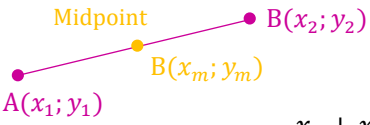
Used to measure the distance(length) between two points.



$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint formula:

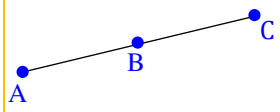
The half – way point between two points.



$$(x_m;\ y_m) = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$$

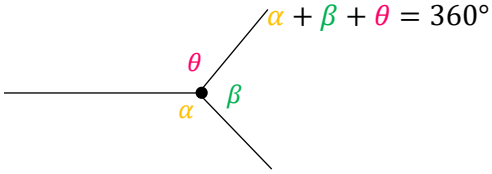
Collinear points:

Points that lie on the same line and have the same gradient.

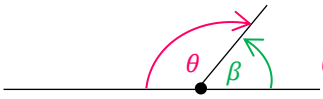


If A, B and C are collinear then the gradient of AB is equal to the gradient of BC and AC.

Angles at a point add up to 360° – **A revolution**

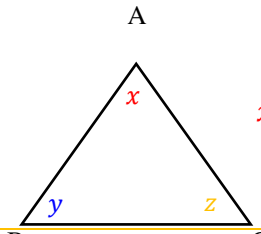


Angles in a straight – line add up to 180°



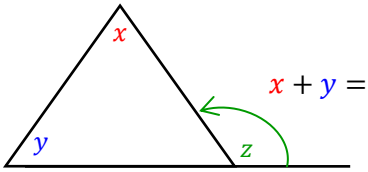
$\theta + \beta = 180°$

Angles in a tringle add up to 180°



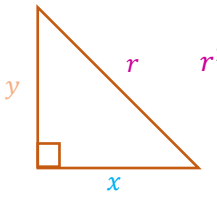
$x + y + z = 180°$

Exterior angle of a triangle



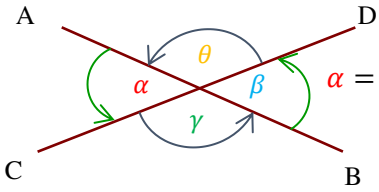
$x + y = z$

Pythagorus Theorem



$r^2 = x^2 + y^2$

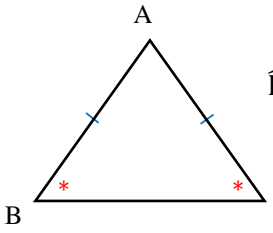
Vertically opposite angles are equal



$\alpha = \beta$ and $\theta = \gamma$

Isosceles triangle:

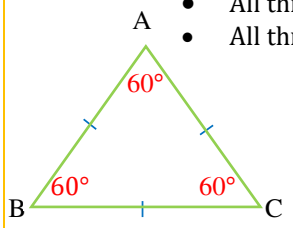
- Two sides are equal
- Base angles are equal



$\hat{B} = \hat{C}$ and AB = AC

Equilateral triangle:

- All three sides are equal
- All three interior angles are equal



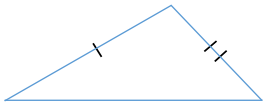
$AB = BC = AC$

$\hat{A} = \hat{B} = \hat{C}$

$\hat{A} + \hat{B} + \hat{C} = 180°$

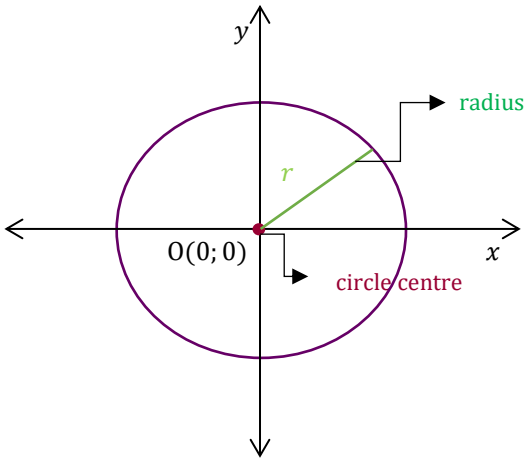
Scalene triangle:

- No sides are equal in length
- No equal angles

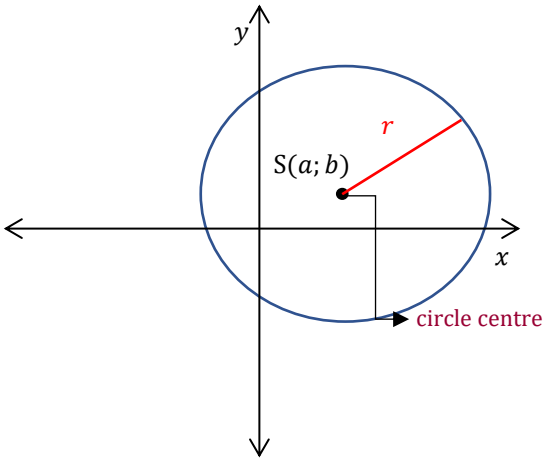


THE CIRCLE

A circle with the centre at the origin (0;0)



A circle with the centre at S $(a; b)$



Equation of the circle : $\boldsymbol{x^2 + y^2 = r^2}$

Equation of the circle: $\boldsymbol{(x - a)^2 + (y - b)^2 = r^2}$

Radius(r): a line drawn from the circle centre to circumference of the circle. Circle centre: $(a; b)$

FINDING THE EQUATION OF THE CIRCLE

Example:1. Determine the equation of the circle with centre $(0; 0)$ and a radius of 5.

Interpretation:

Circle centre (0;0) radius= 5. The equation of a circle with the centre at the origin is: $\boldsymbol{x^2 + y^2 = r^2}$

$\therefore x^2 + y^2 = (5)^2 \rightarrow$ **The equation of the circle is : $\boldsymbol{x^2 + y^2 = 25}$**

Note that the equation: $\boldsymbol{(x - a)^2 + (y - b)^2 = r^2}$ can still be used to find the equation of the circle.

The circle centre (0;0) is $(a; b)$ and $r = 5, \therefore (x - 0)^2 + (y - 0)^2 = (5)^2 \rightarrow \boldsymbol{x^2 + y^2 = 25}$

Example:2. Determine the equation of the circle with centre (0;0) and Q(12;5) a point on the circle.

Interpretation:

Circle centre (0;0) and the radius is the distance between the origin (0;0) and Q (12;5), distance OQ.

$\overset{x_1}{\text{origin}}(\overset{x_1}{0}; \overset{y_1}{0})\ \text{Q}(\overset{x_2}{12}; \overset{y_2}{5}), \text{radius} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \rightarrow \text{radius} = \sqrt{(12 - 0)^2 + (5 - 0)^2} = 13$

Circle centre $(0; 0)$ and radius $= 13$: $\boldsymbol{(x - 0)^2 + (y - 0)^2 = (13)^2}$

$\therefore$ equation of the circle is : $\boldsymbol{x^2 + y^2 = 169}$

SCATTER PLOTS

A scatter plot is a graph that shows the relationship between two variables and their correlation. A scatter plot graph has the independent variable on the $x-axis$ and has the dependent variable on the $y-axis$.

Independent variable: the variable that is being changed.

Dependent variable: the variable that is being measured.

Correlation is the relationship between the independent variable and the dependent variable. **The correlation coefficient is a measure of the strength and direction of the linear relationship between two variables**. The symbol $'r'$ is used to represent the sample correlation coefficient. The range of the correlation coefficient is between -1 and $+1$. If x and y (the dependent and independent variable) have a strong correlation, r is close to 1.If x and y have a negative correlation correlation, r is close to -1. If there is no linear correlation or the linear correlation is weak, r is close to 0.

Types of correlation.

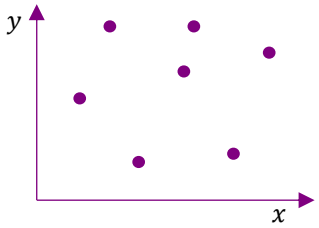
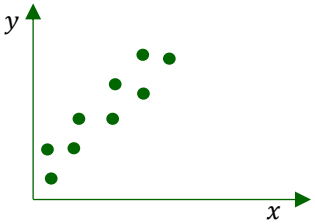
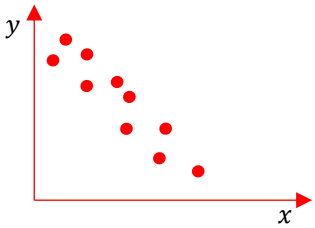
When analyzing correlation between two variables we use these descriptions:

1.The form: weather it is linear or non-linear(either a quadratic or exponential curve).

2.The direction: weather it is positive or negative.

3.The strength: weather it is strong, moderate, or weak.

FORMS OF CORRELATION

ZERO CORRELATION	STRONG POSITIVE CORRELATION	STRONG NEGATIVE CORRELATION
 $r \approx 0$	 $r \approx 0,5$	 $r \approx -0,5$
The points are scattered randomly and there is no correlation or pattern amongst them. ∴ there is **ZERO CORRELATION.**	The points are closely gathered to form a 'band'. The 'band' slopes upward from left to right. ∴ there is a **STRONG POSITIVE CORRELATION.**	The points are closely gathered to form a 'band'. The band slopes downward from left to right. ∴ there is a **STRONG NEGATIVE CORRELATION.**

FUNDAMENTAL COUNTING PRINCIPLE

The fundamental counting principle states that: when there are $\boldsymbol{n}$ ways to make a choice and $\boldsymbol{m}$ ways to make another, then there are $\boldsymbol{n \times m}$ ways to do both.

Factorial Notation: The value of factorial(!) n is found by multiplying together all the whole numbers from 1 up to, and including, n. A special case is $0! = 1$.

Example: $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = \mathbf{5040}$

NOTE

! is accessed by **SHIFT**: $\boldsymbol{x^{-1}}$

There are four(4) rules to consider:

1. **Rule 1:** $\boldsymbol{n \times m}$ → When there are $\boldsymbol{m}$ ways to do one activity and $\boldsymbol{n}$ ways to do another, then there are $\boldsymbol{m \times n}$ ways to do both.

Example:1.Consider the word **DISTINCTION.** How many possible word arrangements can be made if:
a. The letters may be repeated?
b. The letters may not be repeated?

a. The word DISTINCTION has 9 letters, if 9 letters are to be repeated, we use **exponential notation** :$9^9 = \mathbf{387420489}$.
b. Letters are not repeated, ∴we use **factorial notation:** $9! = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362880$.

2. **Rule 2:** $\boldsymbol{n!}$ → When there is an arrangement of $\boldsymbol{n}$ different things taken in $\boldsymbol{n}$ different ways, then it is $\boldsymbol{n!}$.

Example:1.Consider the word **SMART**. How many **six-letter** word arrangements can be made if the letters may **not be repeated**?

Letters may not be repeated,∴ we use the **factorial notation:** $\mathbf{4! = 4 \times 3 \times 2 \times 1 = 24}$.

3. **Rule 3:** $\frac{\boldsymbol{n!}}{\boldsymbol{(n-r)!}}$ → When there is an arrangement of n different things taken in m different ways, then it is $\frac{\boldsymbol{n!}}{\boldsymbol{(n-r)!}}$. **SELECTIONS**

Example:1.Consider the word: **YELLOW.** How many **four-letter** word arrangements can be made if the letters may be repeated?

A four letter arrangement will be:$6 \times 5 \times 4 \times 3$ *or* $\frac{6!}{(6-4)!} = \mathbf{360}$.

4. **Rule 4:** $\frac{\boldsymbol{n!}}{\boldsymbol{m_1! \times m_2! \times m_3! \ldots..}}$ → The number of different ways $\boldsymbol{m_1}$ of the letters are identical,$\boldsymbol{m_2}$ of the letters are identical,$\boldsymbol{m_3}$ of the letters are identical,……$\boldsymbol{m_n}$ of the letters are identical is given by $\frac{\boldsymbol{n!}}{\boldsymbol{m_1! \times m_2! \times m_3! \ldots.}}$, where repeated letters are treated as identical. **REPETITION**